Duration : 60 minutes

Linear Algebra Midterm

Fall 2025

Questions

For the **multiple choice** questions, we give

+3 points if your answer is correct,

0 points if you give no answer or more than one,

−1 if your answer is incorrect.

For the **true/false** questions, we give

+1 points if your answer is correct,

0 points if you give no answer or more than one,

−1 points if your answer is incorrect.

Notation (all standard)

– $\mathbb{R}$ denotes the set of real numbers.
– For a matrix A, $a_{ij} \in \mathbb{R}$ denotes the entry of A in row i and column j.
– For a vector $x \in \mathbb{R}^n$, x_i denotes the ith coordinate of x.
– I_m denotes the $m \times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.
– $\mathbb{R}^{m \times n}$ is the vector space of $m \times n$ matrices.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct response.

Question 1 : The system of linear equations

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 3 \\ x_2 - 2x_3 + 2x_4 = 2 \\ 4x_1 + 3x_2 + 4x_3 - 3x_4 = 1 \\ 5x_1 + 3x_2 + 3x_3 - x_4 = 0 \end{cases}$$

has a unique solution which satisfies

☐ $x_4 = \frac{1}{2}$ ☐ $x_4 = -2$ ☐ $x_4 = 1$ ☐ $x_4 = -\frac{5}{2}$

Question 2 : Let $\underline{b} = \{b_1, b_2, b_3\}$ be an ordered basis of $\mathbb{R}^3$ and let $\underline{c} = \{c_1, c_2, c_3\}$ be an ordered basis of $\mathbb{R}^3$ built by the vectors

$$c_1 = b_1 - b_2 , \quad c_2 = b_1 - b_3 \quad \text{and} \quad c_3 = b_1 + b_2 + b_3 .$$

Let P be the change of basis matrix $[v]_{\underline{c}} = P[v]_{\underline{b}}$ for all $v \in \mathbb{R}^3$. Then

☐ $p_{23} = 0$ ☐ $p_{23} = 1$ ☐ $p_{23} = 2$ ☐ $p_{23} = -\frac{2}{3}$

Question 3 : Let $T \colon \mathbb{R}^{2\times 2} \to \mathbb{P}_2$ be the linear transformation defined by

$$T\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = b_{12} - (b_{11} + b_{22})t + b_{21}t^2 .$$

Let

$$\underline{v} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \right\} \quad \text{and} \quad \underline{w} = \{1,\, 1 + t^2,\, t + t^2\}$$

be ordered bases of $\mathbb{R}^{2\times 2}$ and $\mathbb{P}_2$ respectively. Let A be the matrix that represents T with respect to the bases $\underline{v}$ of $\mathbb{R}^{2\times 2}$ and $\underline{w}$ of $\mathbb{P}_2$, so that $[T(B)]_{\underline{w}} = A[B]_{\underline{v}}$ for all $B \in \mathbb{R}^{2\times 2}$. Then A satisfies

☐ $a_{12} = 3$ ☐ $a_{12} = -4$ ☐ $a_{12} = -6$ ☐ $a_{12} = 2$

Question 4 : Let $\alpha \in \mathbb{R}$ and $B \in \mathbb{R}^{4\times 4}$ defined by

$$B = \begin{pmatrix} 1 & 1 & \alpha & 0 \\ 1 & 1 & 0 & \alpha \\ \alpha & 0 & 1 & 1 \\ 0 & \alpha & 1 & 1 \end{pmatrix} .$$

Then

☐ $\det(B) = \alpha^2(\alpha^2 - 4)$ ☐ $\det(B) = \alpha^2(\alpha - 1)^2$
☐ $\det(B) = \alpha(\alpha + 2)(\alpha - 1)^2$ ☐ $\det(B) = \alpha(\alpha + 1)(\alpha^2 - 4)$

Question 5 : Let $v = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ and $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ the linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y - 5z \\ -2x + y + 7z \\ x - 2z \end{pmatrix}.$$

Then

- ☐ the vector v is in $\mathrm{Im}(T)$ and in $\mathrm{Ker}(T)$
- ☐ the vector v is in $\mathrm{Ker}(T)$, but not in $\mathrm{Im}(T)$
- ☐ the vector v is in $\mathrm{Im}(T)$, but not in $\mathrm{Ker}(T)$
- ☐ the vector v is neither in $\mathrm{Ker}(T)$, nor in $\mathrm{Im}(T)$

Question 6 : Let a be a real number. The linear system

$$\begin{cases} a\,x_1 \quad + \quad x_3 = 1 \\ a\,x_2 \quad + \quad x_4 = 0 \\ x_1 \quad + a\,x_3 = 1 \\ x_2 \quad + a\,x_4 = 1 \\ x_1 \quad + \quad x_3 = a \end{cases}$$

has exactly one solution if and only if

☐ $a \in \{-1, 1, -2\}$ ☐ $a = -2$ ☐ $a \in \{-1, 1\}$ ☐ $a \in \{-2, 1\}$

Question 7 : Let

$$A = \begin{pmatrix} \pi & 1 & 3 & 1 \\ 2\pi & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

If $B = A^{-1}$ is the inverse of the matrix A, then

☐ $b_{24} = 3$ ☐ $b_{22} = 1$ ☐ $b_{23} = 5$ ☐ $b_{21} = 7$

Question 8 : Let W be the vector space of symmetric matrices of size $2{\times}2$. The second coordinate of the element $\begin{pmatrix} -3 & 2 \\ 2 & 1 \end{pmatrix} \in W$ with respect to the ordered basis $\left\{ \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right\}$ of W is

☐ -2 ☐ 1 ☐ $-\frac{3}{4}$ ☐ $\frac{1}{4}$

Second part: true/false questions

For each question, mark the box TRUE if the statement is **always true** or the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 9: Let A be a matrix of size $m\times n$ and let B be a matrix of size $n\times m$. If AB is invertible, then both A and B are also invertible.

☐ TRUE ☐ FALSE

Question 10: The number of non-zero rows in the reduced echelon form of a matrix is equal to the rank of the matrix.

☐ TRUE ☐ FALSE

Question 11: Let V and W be two vector spaces and let $T, S\colon V \to W$ be two linear transformations. If $\text{Ker}(T) = \text{Ker}(S)$, then $\text{Im}(T) = \text{Im}(S)$.

☐ TRUE ☐ FALSE

Question 12: The set

$$\{p \in \mathbb{P}_n : p(1)\, p(2) = 0\}$$

is a subspace of $\mathbb{P}_n$.

☐ TRUE ☐ FALSE

Question 13: Let V be a vector space and let v_1, v_2, v_3 be three vectors in V. If the set $\{v_1\,, v_2\,, v_3\}$ is linearly independent, then the set $\{v_1 - v_2\,, v_1 - v_3\,, v_2 - v_3\}$ is linearly independent as well.

☐ TRUE ☐ FALSE

Question 14: Let A and B be two matrices of size 3×3. If $\det(A) = 2$, then

$$\det(BA^{-1}B) = \frac{1}{2}\det(B)^2\,.$$

☐ TRUE ☐ FALSE

Question 15: If $A \in \mathbb{R}^{m\times n}$ satisfies $\text{Col}(A) = \mathbb{R}^m$, then for every $b \in \mathbb{R}^m$ the linear system $Ax = b$ has a unique solution.

☐ TRUE ☐ FALSE

Question 16: Let V be a vector space and let $T\colon V \to \mathbb{R}$ be a linear transformation. If v_1, v_2, $v_3 \in V$ are so that

$$T(v_1) = 2\,, \quad T(v_2) = 3 \quad \text{and} \quad T(v_3) = 1\,,$$

then $v_1 - v_2 + v_3$ is in $\text{Ker}(T)$.

☐ TRUE ☐ FALSE