MATH-111(en) FALL 2025
Linear Algebra

developed by Annina Iseli
taught by Andrei Negut

MINI SOLUTIONS for Homework 13

Ex 13.1 (Using the Gram–Schmidt process)
Let W be the subspace of $\mathbb{R}^4$ spanned by the basis vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix}.$$

a) Construct an orthogonal basis for W using the Gram–Schmidt process.

b) Consider $A = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3]$ having the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ as columns. Find out a QR decomposition of A.

Solution:

a)

$$\mathbf{v}_1 = \mathbf{x}_1, \mathbf{v}_2 = \begin{pmatrix} 3/2 \\ 3/2 \\ -3/2 \\ -3/2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$$

b)

$$Q = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{10} \\ -1/2 & 1/2 & 2/\sqrt{10} \\ -1/2 & -1/2 & 1/\sqrt{10} \\ 1/2 & -1/2 & 2/\sqrt{10} \end{pmatrix} \text{ and } R = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & \sqrt{10} \end{pmatrix}.$$

Ex 13.2 (Finding an orthonormal basis)
Find an orthonormal basis for the span of the following vectors.

$$\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$$

Solution:

$\left\{ \frac{1}{\sqrt{50}} \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \right\}$ forms an orthonormal basis of the span of the given vectors.

Ex 13.3 (QR factorization)
Find a QR factorization for each of the following matrices:

$$A = \begin{pmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{pmatrix}$$

Solution:
<u>Factorization for A</u>:

$$Q = \frac{1}{7}\begin{pmatrix} -2 & 5 \\ 5 & 2 \\ 2 & -4 \\ 4 & 2 \end{pmatrix} \text{ and } R = \begin{pmatrix} 7 & 7 \\ 0 & 7 \end{pmatrix}.$$

<u>Factorization for B</u>:

$$Q = \frac{1}{\sqrt{12}}\begin{pmatrix} -1 & 3 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{pmatrix} \text{ and } R = \frac{1}{\sqrt{3}}\begin{pmatrix} 6 & -18 & 3 \\ 0 & 6 & 15 \\ 0 & 0 & 6 \end{pmatrix}.$$

Ex 13.5 (A least-squares problem)
Find all least-squares solution $x^\star$ of the system $Ax = b$ and their least square errors $\|Ax^\star - b\|$.

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

Solution:
$\mathbf{x}^* = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ is the only solution.

Ex 13.6 (Another least-squares problem)
Find all least-squares solution $x^\star$ of the system $Ax = b$ and their least square errors $\|Ax^\star - b\|$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

Solution:
$\mathbf{x}^* = \begin{pmatrix} -t+5 \\ t-3 \\ t \end{pmatrix}$ for any $t \in \mathbb{R}$ are the least-square solutions with corresponding least square $\sqrt{20}$.

Ex 13.7 (QR decomposition for a least-square problem)
Consider

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}.$$

a) Show that

$$A = \begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}.$$

b) Use this QR decomposition of A to find the least squares solution to the equation $A\mathbf{x} = b$.

Solution:

a) Omitted.

b)

$$\mathbf{x}^* = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

Ex 13.8 (Linear regression)

(a) Find the straight line that best approximates (in the sense of least squares) the following data points in $\mathbb{R}^2$: $(2,1), \quad (5,2), \quad (7,3), \quad (8,3)$

(b) Draw a picture that illustrates the data points and the line that best approximates them.

Solution:
The line that fits best is $y = \frac{5}{14}x + \frac{4}{14}$ or $5x - 14y = -4$.

Ex 13.9 (Linear regression)
Assume that you measure the measure the temperature near a chemical experiment at times $t = 1, 2, 3, 4, 5, 6$. The measurements y (ordered by time) that you obtain are $20, 30, 35, 40, 45, 45$. Find a linear function $f(t) = y$ approximating your data with minimal least square error. Also, give the value of the least square error.

Solution: The linear function approximating the data with minimal least square error is

$$f(t) = 5t + 18 + \frac{1}{3}$$

and the least square error for f is $\frac{10}{\sqrt{3}}$.