MATH-111(en) FALL 2025
Linear Algebra

developed by Annina Iseli
taught by Andrei Negut

MINI SOLUTIONS for Homework 12

Ex 12.2 (Inner product calculations)
Let

$$u = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, \qquad v = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}.$$

a) Calculate $u \cdot u$, $v \cdot v$, $u \cdot v$, $\|u\|$, and $\|v\|$.

b) Normalize u and v (i.e., find a unit vector with the same direction).

c) Find the distance between u and v, and find the cosine of the angle between them.

d) Find a basis of the space orthogonal to the plane spanned by u and v.

Solution:

a) $u \cdot u = 35$, $v \cdot v = 49$, $u \cdot v = 35$, $\|u\| = \sqrt{35}$, $\|v\| = 7$.

b) $\frac{u}{\|u\|} = \frac{1}{\sqrt{35}} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$, $\frac{v}{\|v\|} = \frac{1}{7} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$

c) The distance between u and v is

$$\|u - v\| = \sqrt{14}$$

and the cosine of the angle is

$$\cos(\alpha(u, v)) = \frac{\sqrt{35}}{7} \left(= \sqrt{\frac{5}{7}} \right).$$

d) A basis of the space is

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

Ex 12.3 (An orthogonal basis)
Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}, \qquad u = \begin{pmatrix} 10 \\ 4 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

Show that $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^3$ and determine $[u]_{\mathcal{B}}$ and $[v]_{\mathcal{B}}$, i.e. represent them in the basis $\mathcal{B}$.

Solution:

$$[u]_\mathcal{B} = \tfrac{1}{6}\begin{pmatrix} -3 \\ 34 \\ -13 \end{pmatrix}, \quad [v]_\mathcal{B} = \tfrac{1}{6}\begin{pmatrix} 15 \\ 4 \\ -1 \end{pmatrix}$$

Ex 12.4 (Another orthogonal basis)
Consider the vectors

$$u = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \quad x = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

(a) Show that $\{u, v, w\}$ is an orthogonal basis of $\mathbb{R}^3$.

(b) Write the vector x as a linear combination of u, v and w.

Solution:

(a) Omitted.

(b) $x = \frac{4}{3}u + \frac{1}{3}v + \frac{1}{3}w$.

Ex 12.6 (F^TF vs. FF^T for matrices with orthogonal columns)
Consider the matrix

$$F = \begin{pmatrix} 1 & 2 \\ -4 & 1/2 \end{pmatrix}.$$

Compute F^TF and FF^T. Are these two matrices equal ?

Solution:

$$F^TF = \begin{pmatrix} 17 & 0 \\ 0 & 17/4 \end{pmatrix}, \quad FF^T = \begin{pmatrix} 5 & -3 \\ -3 & 65/4 \end{pmatrix}.$$

Ex 12.8 (Projection onto a subspace)
Let

$$u = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Determine the orthogonal projection $\text{proj}_W(u)$ of u onto the subspace W spanned by v_1, v_2. Give it both in the basis $\mathcal{B} = \{v_1, v_2\}$ of W and in the standard basis of $\mathbb{R}^3$.

Solution:
$\text{proj}_W(u) = \frac{1}{2}v_1 + \frac{6}{3}v_2$

an denoting the standard basis by $\mathcal{E}$, we have: $[\text{proj}_W(u)]_\mathcal{B} = \begin{pmatrix} 1/2 \\ 2 \end{pmatrix}$, $[\text{proj}_W(u)]_\mathcal{E} = \frac{1}{2}\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$.

Ex 12.10 (Closest point in a column space)
Let A be the following matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

1. Show that the columns of A are an orthogonal set.
2. Write U, the matrix made of the normalized columns vectors of A.
3. Find the closest point to $y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ in $\text{Col}(U)$ and the distance from $b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ to $\text{Col}(U)$.

Solution:

1. Omitted.
2. $$U = \begin{pmatrix} 1/\sqrt{2} & -1/2 & 1/\sqrt{6} \\ 0 & 1/2 & 2/\sqrt{6} \\ -1/\sqrt{2} & -1/2 & 1/\sqrt{6} \\ 0 & 1/2 & 0 \end{pmatrix}.$$
3. Denoting t he closest point to y in $\text{Col}(U)$ as $\hat{y}$, we have
$$\hat{y} = \begin{pmatrix} 2/3 \\ 4/3 \\ 2/3 \\ 0 \end{pmatrix}.$$
The distance from b to $\text{Col}(U)$ is $\sqrt{3}$.

Ex 12.11 (Distance to different subspaces)

Let $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$. Calculate the distance from u to the line spanned by v_1, and the distance from u to the plane spanned by v_1 and v_2.

Solution:
The distance from u to the line spanned by v_1 is $\frac{\sqrt{30}}{5}$ while the distance from u to the plane spanned by v_1 and v_2 is $\frac{7}{3\sqrt{5}}$.