MATH-111(en) FALL 2025
Linear Algebra
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MINI SOLUTIONS for Homework 11

Ex 11.2 (Diagonalization of a matrix)

Diagonalize the following matrix.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

If not stated otherwise, this means finding a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. In particular, you do not need to compute P^{-1}.

Solution:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, P = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Ex 11.3 (Déjà vu?)

Diagonalize the following matrix.

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution:

$$P = \begin{pmatrix} 2 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Ex 11.4 (More diagonalizability examples)

Consider

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}, \; B = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \; C = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}, \; D = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \quad \text{and } E = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}.$$

1. For each matrix find out the eigenvalues and the corresponding eigenvectors.
   Hint: For C and D the rational root theorem (see Ex. 10.8) helps to find the eigenvalues.
2. Find out which ones are diagonalizable.

Solution:

- A has unique eigenvalue $\lambda = 4$ with eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

- B has eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 5$ with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

- C has the following eigenvalue/eigenvector pairs

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for } \lambda = 1, \quad \begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix} \text{ for } \lambda = 2, \quad \begin{pmatrix} 1/4 \\ 3/4 \\ 1 \end{pmatrix} \text{ for } \lambda = 3.$$

- D has the following eigenvalue/eigenvector pairs

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for } \lambda = 8. \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ for } \lambda = 2.$$

- E has one eigenvalue 5 with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Ex 11.6 (Matrix representation of linear maps)
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 4z \\ 3x + 5y - 2z \\ x + y + 4z \end{pmatrix}.$$

Consider the ordered basis of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Find the matrix $M = \underset{\mathcal{B} \leftarrow \mathcal{B}}{[T]}$ that represents T in the basis $\mathcal{B}$.

Solution:

$$\underset{\mathcal{B} \leftarrow \mathcal{B}}{[T]} = \begin{pmatrix} 6 & 2 & 1 \\ 0 & 6 & 2 \\ -2 & -8 & -3 \end{pmatrix}$$

Ex 11.7 (Another matrix representation)
Let the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined by:

$$T(\mathbf{p}) = \begin{pmatrix} p(0) \\ p(0) \\ p(2) \end{pmatrix} \quad \text{for any polynomial } \mathbf{p} \in \mathbb{P}_2.$$

a) Find the matrix A of the linear transformation T in terms of the standard bases of $\mathbb{P}_2$ and $\mathbb{R}^3$.

b) Using the matrix A, determine the kernel and image of T.

Solution:

a)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix}.$$

b)

$$\operatorname{Ker} T = \{\mathbf{p} \in \mathbb{P}_2 : \quad p(t) = \alpha \cdot (-2t + t^2) \quad \text{with} \quad \alpha \in \mathbb{R}\}$$

and

$$\operatorname{Ran} T = \left\{ \mathbf{x} \in \mathbb{R}^3 : \quad \mathbf{x} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \quad \text{with} \quad \alpha \in \mathbb{R} \quad \text{and} \quad \beta \in \mathbb{R} \right\}.$$