**MATH-111(en)** FALL 2025
**Linear Algebra**
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# MINI SOLUTIONS for Homework 8

**Ex 8.1 (A family of bases)**
Find all $b \in \mathbb{R}$ such that the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ b \\ 0 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$.

**Solution:** $B = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ if and only if $b \neq 4$.

**Ex 8.2 (Basis or not?)**
Determine if

$$\{1 + t^2, 1 - t, 2 - 4t + t^2, 6 - 18t + 9t^2 - t^3\}$$

is a basis for $\mathbb{P}_3 = \{\text{degree } \leq 3 \text{ polynomials in } t\}$.

**Solution:** They do form a basis.

**Ex 8.3 (Bases of column and null spaces)**
Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

(a) Find a basis for the column space of $A$.

(b) Find a basis for the null space of $A$.

**Solution:** Many solutions are possible as the basis of a vector space is not unique. Though by using the technique shown in class, you likely ended up finding the following solution:

(a)

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix} \right\} \text{ is a basis of } \mathrm{Col}(A).$$

(b)

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{is a basis of } \mathrm{Ker}(A).$$

**Ex 8.7 (Representing a vector in a different basis)**
Let $\underline{b} = \{b_1, b_2, b_3\}$ be the basis of $\mathbb{R}^3$ with

$$b_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the vector $u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, determine $[u]_{\underline{b}}$.

Moreover, find the vector $w$ such that $[w]_{\underline{b}} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

**Solution:**

$$[u]_{\underline{b}} = \frac{1}{4} \begin{pmatrix} 5 \\ 3 \\ -9 \end{pmatrix}, \qquad w = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}.$$

**Ex 8.9 (New coordinates for polynomials)**
Consider the basis $\underline{b} = \{p_1, p_2, p_3\}$ of $\mathbb{P}_2$ with

$$p_1(t) = 1 + t + t^2, \quad p_2(t) = 2t - t^2, \quad p_3(t) = 2 + t - t^2.$$

Determine $[t]_{\underline{b}}$ and $[1 + t^2]_{\underline{b}}$.
**Hint**: Write the $\underline{b}$-coordinates of $p_1$, $p_2$ and $p_3$ and the polynomials $t$ and $1+t^2$ for the basis $\underline{b} = \{1, t, t^2\}$ and then solve the corresponding linear systems.

**Solution:**

$$[t]_{\underline{b}} = \frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \qquad [1 + t^2]_{\underline{b}} = \frac{1}{7} \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$