

MATH-111(en)  
Linear Algebra

FALL 2025  
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## MINI SOLUTIONS for Homework 7

### Ex 7.2 (Is it a vector space?)

For each of the following sets (equipped with the obvious addition and scalar multiplication), decide whether it is a vector space and prove your result.

$$A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = 0 \right\}, \quad B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y = 1 \right\}, \quad C = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : z = y \right\}$$

$$D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \{0, -1, 1\} \right\}, \quad E = \{f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ linear} : f(e_1) = 0\}$$

**Solution:** A, C, and E are vector spaces. B and D are not.

### Ex 7.3 (Spaces of polynomials)

Let  $P_n$  be the vector space of polynomials of degree less than or equal to  $n$ . Determine if each of the following sets is a subspace of  $P_n$  for a given  $n$ . (You may take for granted that  $P_n$  is a vector space.)

- The set of polynomials of the form  $p(t) = at^2$  where  $a$  is an arbitrary real number.
- The set of polynomials of the form  $p(t) = a + t^2$  where  $a$  is an arbitrary real number.
- The set of polynomials of the form  $p(t) = c_1t^3 + c_2t^2 + c_3t + c_4$ , where  $c_1, c_2, c_3$  and  $c_4$  are non-negative integers.
- The set of polynomials in  $P_n$  that satisfy  $p(0) = 0$ .

**Solution:** (a) and (d) are subspaces. (b) and (c) are not.

### Ex 7.6 (The only finite subspaces is $\{0_v\}$ .)

Let  $V$  be a vector space and  $0_V$  its zero element. Prove that  $\{0_V\}$  is the only subspace of  $V$  that consists of only finitely many elements.

#### Hint:

Let  $H$  be a subspace of  $V$  that is not equal to  $\{0_v\}$ . Let  $x \in H$  be a vector that is not  $0_V$ . Try to prove the following claim and explain, why it solves Exercise 7.6.

Claim: Let  $k_1, k_2 \in \mathbb{R}$ . If  $k_1 \neq k_2$  then  $k_1x \neq k_2x$ .

(In other words: for each  $k$ ,  $kx$  is a different vector in  $H$ .)

**Ex 7.7 (Column space and kernel)**

- (a) Does  $v = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$  lie in the column space of  $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{pmatrix}$ ? Does it lie in its kernel?
- (b) Let  $B = \begin{pmatrix} 1 & 2 & -3 \\ 4 & -1 & 0 \\ 0 & -3 & 4 \end{pmatrix}$ . Find a nonzero vector  $u \in \text{Col}(B)$  and a nonzero vector  $v \in \text{Ker}(B)$ . Is there a nonzero vector that lies in both  $\text{Col}(B)$  and  $\text{Ker}(B)$ ?
- (c) Express the kernel of the following matrix in parametric vector form:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

**Solution:**

- (a)  $v$  lies in the column space but not the kernel.
- (b) Omitted.
- (c)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s \\ -t \\ s \\ t \end{pmatrix} = s \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

**Ex 7.8 (Column space and kernel)**

- (a) Consider

$$w = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 3 & -5/2 \\ -3 & -2 & 4 \\ 2 & 4 & -4 \end{pmatrix}.$$

Find out if  $w$  is in  $\text{Col } A$ , in  $\text{Ker } A$ , or both.

- (b) Find bases for the kernel, the column space, and the row space of  $A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 2 & 4 & 14 & 4 \\ 2 & 3 & 12 & 3 \end{pmatrix}$

**Solution:**

- (a):  $w$  is in  $\text{Ker } A$  as well as in  $\text{Col } A$ .

(b): If you follow the steps of Example ( $\star$ ) from class (Week 7). You will find obtain the following bases:

$$\mathcal{B}_{\text{Ker}(A)} = \left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \mathcal{B}_{\text{Col}(A)} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\}, \quad \mathcal{B}_{\text{Row}(A)} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\},$$

Please be aware that different solutions are possible as no vector space has a unique basis.

If you need a repetition of the methods, the following youtube video goes over this exact example: <https://www.youtube.com/watch?v=AVWT1NkTNfw>