MATH-111(en) FALL 2025
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MINI SOLUTIONS for Homework 2

Ex 2.1 (The weekly linear system)
Solve the following linear system and write the solution in parametric form.

$$\begin{cases} x + 2y + 2t = 3 \\ y + z - 2t = 0 \\ x + 3y + z + t = 5 \\ 2x + 5y + z + 4t = 10 \end{cases}$$

Solution :
The solution in parametric vector form is

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -9+2z \\ 4-z \\ z \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ 4 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

Ex 2.2 (On $B(Ax) \neq A(Bx)$)
We first observe that : If $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$, then $Ax \in \mathbb{R}^m$ is well-defined. If $B \in \mathbb{R}^{p\times m}$, then the element $B(Ax) \in \mathbb{R}^p$ is well-defined.

a) Determine all values of m and p (depending on n) such that $A(Bx)$ is also defined.

b) For $n = 2$, consider the matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Show that for $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ it holds that $B(Ax) \neq A(Bx)$. Find all vectors $x \in \mathbb{R}^2$ such that $B(Ax) = A(Bx)$.

Solution :
a) $m = p = n$.

b) They agree only for $x = 0$.

Ex 2.3 (Conversion to parametric vector form)
Write all solutions to the linear systems from Problems 1, 2, and 3 from Homework 1 in parametric vector form.

Solution :

1. (a)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{29}{7} \\ \frac{6}{7} \end{pmatrix}.$$

(b)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -46 \\ -46 \\ 95 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} \frac{-4}{5} \\ \frac{1}{5} \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{3}{5} \\ \frac{-2}{5} \\ 0 \end{pmatrix} \text{ where } s \in \mathbb{R}.$$

2. (a)

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{7}{5} \\ \frac{2}{5} \\ \frac{1}{5} \\ 0 \end{pmatrix} \text{ where } s \in \mathbb{R}.$$

(b) No solution.

(c)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ where } s, t \in \mathbb{R}.$$

3. (a)

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ where } s, t \in \mathbb{R}.$$

(b)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Ex 2.4 (A system with a parameter)
For

$$A = \begin{pmatrix} 1 & -1 & -3 \\ 2 & 0 & 1 \\ 0 & 2 & 7 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ a \end{pmatrix},$$

find an a such that $A\mathbf{x} = \mathbf{b}$ is consistent, and write the solution in this case in parametric vector form.

Solution :
The matrix equation has a solution if and only if $a = -6$. In this case the solution in parametric vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} == \lambda \cdot \begin{pmatrix} -1/2 \\ -7/2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

Ex 2.5 (Consistency for all right-hand sides ?)
Let

$$\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}.$$

Is the equation $\mathbf{Ax} = \mathbf{b}$ consistent for all choices of $\mathbf{b} \in \mathbb{R}^2$? Determine the set of $\mathbf{b} \in \mathbb{R}^2$ for which the equation $\mathbf{Ax} = \mathbf{b}$ is consistent.

Solution :
The set of $\mathbf{b}$ for which $\mathbf{Ax} = \mathbf{b}$ is consistent is

$$\left\{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, 2b_1 + b_2 = 0 \right\}.$$

Ex 2.6 (Homogeneous and inhomogeneous systems)

Consider the two systems

$$\begin{cases} x_1 - 3x_2 - 2x_3 &= 0 \\ x_2 - x_3 &= 0 \\ -2x_1 + 3x_2 + 7x_3 &= 0 \end{cases} \qquad \begin{cases} x_1 - 3x_2 - 2x_3 &= -5 \\ x_2 - x_3 &= 4 \\ -2x_1 + 3x_2 + 7x_3 &= -2 \end{cases}.$$

For each of these systems, write the solution in its parametric vector form and give a geometric description of the solution space. For example, say what "shape" it is and whether or not it contains the origin.

Solution :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} + s \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

Ex 2.7 (A homogeneous equation)

For the following matrix $\mathbf{A}$, write the solution of $\mathbf{Ax} = \mathbf{0}$ in its parametric vector form.

$$\mathbf{A} = \begin{pmatrix} 1 & 6 & 0 & 8 & -1 & -2 \\ 0 & 0 & 1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = s \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -8 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Ex 2.9 (Two related systems)

Solve the following two systems of linear equations and write the solutions in parametric vector form. What is the connection between the solution sets of the two systems ?

$$\begin{cases} x + y - 3z &= 0 \\ 3x + 7y - 13z &= 0 \\ x - y - z &= 0 \end{cases} \qquad \begin{cases} x + y - 3z &= 0 \\ 3x + 7y - 13z &= -4 \\ x - y - z &= 2 \end{cases}$$

Solution :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$