

## MINI SOLUTIONS for Homework 1

Mini solutions will only contain questions with numerical solutions. Namely, it will not include solutions to True/False, multiple choice nor *show that* type of questions.

**Ex 1.1 (Solving linear systems I)**

Solve the following systems of linear equations by finding the augmented matrix and using row elimination.

$$a) \begin{cases} x + y = 5 \\ 2x - 5y = 4 \end{cases} \quad b) \begin{cases} x + y + z = 3 \\ x - y = 0 \\ x = -46 \end{cases} \quad c) \begin{cases} 2x + 3y + z = 0 \\ x - y + z = 1 \\ 3x + 2y + 2z = 1 \end{cases}$$

**Solution :**

a) 
$$x = \frac{29}{7}, \quad y = \frac{6}{7}$$

b) 
$$\rightarrow x = -46, \quad y = -46, \quad z = 95$$

c) There are infinitely many solutions with the solution space being

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = -\frac{4}{5}z + \frac{3}{5}, \quad y = \frac{1}{5}z - \frac{2}{5}, \quad z \in \mathbb{R} \right\}$$

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**Ex 1.2 (Solving linear systems II)**

Find the augmented matrix of the following linear systems and use it to solve them.

$$a) \begin{cases} w - x + z = 1 \\ x - 2y - z = 0 \\ w + 3y = 2 \end{cases} \quad b) \begin{cases} 2x - 5y + 4z = 0 \\ x + y + z = 0 \\ 4x - 3y + 6z = 1 \end{cases} \quad c) \begin{cases} x - 2y + 3z = 1 \\ 2x - 4y + 6z = 2 \\ -x + 2y - 3z = -1 \end{cases}$$

**Solution :**

a) The augmented matrix is

$$\left( \begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 & 0 \\ 1 & 0 & 3 & 0 & 2 \end{array} \right)$$

with the solution space being

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y = \frac{1}{5}, \quad x = \frac{2}{5} + z, \quad w = \frac{7}{5}, \quad z \in \mathbb{R} \right\}$$

b) The augmented matrix is

$$\left( \begin{array}{ccc|c} 2 & -5 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 4 & -3 & 6 & 1 \end{array} \right)$$

and there are no solutions.

c)

$$\left( \begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & 2 \\ -1 & 2 & -3 & -1 \end{array} \right)$$

and the solutions space is

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = 2y - 3z + 1, \quad y, z \in \mathbb{R} \right\}$$

### Ex 1.3 (Solving linear systems III)

Find the augmented matrix of the following linear systems and use it to solve them.

$$a) \begin{cases} w - x + y - z = 1 \\ w + z = 2 \end{cases} \quad b) \begin{cases} x + y = 0 \\ 3x + 5y = 2 \\ 2x + 4y = 2 \end{cases}$$

**Solution :**

a) The augmented matrix is

$$\left( \begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 2 \end{array} \right)$$

and the solutions space is

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x = y - 2z + 1, \quad w = -z + 2, \quad y, z \in \mathbb{R} \right\}$$

b) The augmented matrix is

$$\left( \begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 5 & 2 \\ 2 & 4 & 2 \end{array} \right)$$

and the solutions space is

$$\mathcal{S} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

### Ex 1.4 (Linear systems with a parameter)

Determine for which  $a \in \mathbb{R}$  the following system has no solution, a unique solution, or infinitely many solutions.

$$\begin{cases} x - 2y + 3z = 2 \\ x + 3y - 2z = 5 \\ 2x - y + az = 1 \end{cases}$$

**Solution :**

$$\begin{cases} a = 3 : & \text{no solutions} \\ a \neq 3 : & \text{unique solution : } x = \frac{16}{5} + \frac{24}{5a-15}, \quad y = \frac{3}{5} - \frac{24}{5a-15}, \quad z = \frac{-24}{5a-15} \end{cases}$$

**Ex 1.5 (Solvability of parameter-dependent systems)**

Determine the values of  $h$  for which the following matrices are the augmented matrices of a consistent linear system. (A linear system is called *consistent* if it has at least one solution. It is *inconsistent* if there exists no solution.)

$$(a) \begin{pmatrix} 1 & -3 & h \\ -2 & 6 & -5 \end{pmatrix}, \quad (b) \begin{pmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{pmatrix}.$$

**Solution :**

a) It is consistent if and only if  $h = 5/2$ .

b) It is consistent if and only if  $h \neq 2$ .

**Ex 1.6 (Linear combinations)**

a) For the vectors  $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\mathbf{c} = \begin{pmatrix} 11 \\ 16 \\ 21 \end{pmatrix}$ , find  $l, m \in \mathbb{R}$  such that  $\mathbf{c} = l\mathbf{a} + m\mathbf{b}$ .

b) Find all  $a \in \mathbb{R}$  such that  $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ 2 \end{pmatrix} \right\}$ .

**Solution :**

a)  $l = 3, m = 2$

b)  $a = 2$  or  $a = -1$ .