

MATH-111(en)
Linear Algebra

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developed by Annina Iseli
taught by Andrei Negut

Homework 6

Ex 6.1 (The determinant of triangular matrices)

Let $A \in \mathbb{R}^{n \times n}$ be a lower (resp. upper) triangular matrix. Show that the determinant of A equals the product of its diagonal elements, i.e., $\det(A) = a_{11} \cdot \dots \cdot a_{nn} = \prod_{i=1}^n a_{ii}$.

Hint: Lower triangular A : use the definition of the determinant and induction on n . Upper triangular A : use that $\det(A^T) = \det(A)$.

Ex 6.2 (Row reduction and the determinant)

Let $A \in \mathbb{R}^{n \times n}$ and assume that A_1 is obtained from A by applying an elementary operation on the rows of A . Show that

$$\det(A) = \begin{cases} -\det(A_1) & \text{if two rows or columns have been swapped,} \\ \lambda^{-1} \det(A_1) & \text{if a row or column has been multiplied by } \lambda \neq 0, \\ \det(A_1) & \text{if a multiple of a row (resp. column) has been added to another row (resp. column)} \end{cases}$$

Hint: We know that $A_1 = EA$ for some elementary matrix E .

Ex 6.3 (Different methods for computing determinants)

Compute the determinant of each of the following matrices in three ways: Once using cofactor expansion across a row, once using cofactor across a column, and once with row reduction.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 8 & 0 \end{pmatrix}$$

Ex 6.4 (Determinants based on another determinant)

Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and assume that $\det(A) = 7$. Compute the determinants of the following matrices

$$B = \begin{pmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{pmatrix}, \quad C = \begin{pmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{pmatrix}.$$

Ex 6.5 (More determinants)

Compute the determinants of the following matrices (You may use your preferred method or

try to practice different methods.)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 10 & 5 & 10 & 5 \\ 6 & 9 & 0 & -3 \\ 3 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Ex 6.6 (Determinant of an antidiagonal matrix)

Find the determinant of the following *antidiagonal* matrix:

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$$

What is the determinant of an $n \times n$ antidiagonal matrix with all its antidiagonal entries equal to 2?

Ex 6.7 (Determinants and volume)

(a) Calculate the volume of the parallelepiped with the following vertices:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

(b) Calculate the area of the triangle whose vertices are the points $(1, 2), (2, 4), (3, 3) \in \mathbb{R}^2$.

Ex 6.8 (Abstract determinant calculations)

Show that

- (a) if A is an invertible matrix, then $\det(A^{-1}) = 1/\det A$,
- (b) if A and P are square matrices, with P invertible, then $\det(PAP^{-1}) = \det A$,
- (c) if U is a square matrix such that $U^T U = I$, then $\det U = \pm 1$,
- (d) if A is a square matrix such that $\det(A^4) = 0$, then A cannot be invertible.

Ex 6.9 (Multiple choice and True/False questions)

- a) Let A be an $n \times n$ matrix with nonzero determinant. Then $\det(A + A) =$
(A) 0 (B) $\det(A)$ (C) $2 \det(A)$ (D) $2^n \det(A)$.
- b) In the following, we assume that all the matrices involved are square matrices. Decide whether the following statements are always true or if they can be false.
 - (i) The following matrix is invertible.

$$\begin{pmatrix} 0 & 0 & 0 & 4 \\ 2 & 0 & 4 & 3 \\ 1 & 0 & 2 & 5 \\ 3 & 6 & 1 & 8 \end{pmatrix}$$

- (ii) The (i, j) -cofactor of a square matrix A is the matrix A_{ij} obtained by deleting from A its i -th row and j -th column.
- (iii) If A and B are both $n \times n$, then $\det(A + B) = \det(A) + \det(B)$.
- (iv) If A and B are row equivalent, i.e., they can be obtained from each other by finitely many elementary operations, then they have the same determinant.
- (v) The linear transformation associated to A is injective if and only if $\det(A) \neq 0$.
- (vi) The determinant of A is the product of the pivots in any echelon form U of A , multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U .
- (vii) The determinant of A is the product of the diagonal entries in A .
- (viii) If $\det A$ is zero, then two rows or two columns are the same, or a row a column is zero.