MATH-111(en) FALL 2025
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Homework 5

Do NOT use determinants to solve the problems on this homework assignment. Next weeks exercises will be full of problems about determinants. This week, we practice other methods.

Ex 5.1 (Elementary matrix or not?)

Which of the following matrices are elementary matrices and what elementary operations do they represent?

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hint: Try either (or both) of the following strategies: (1) compare each matrix with the definitions of the three types of elementary matrices that we have seen in class; (2) Compute the matrix-matrix product of each of the matrix with another matrix.

Ex 5.2 (Different methods for computing the inverse matrix)

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Compute A^{-1} using $\begin{cases} \text{(a) the formula for the inverse of a } 2 \times 2 \text{ matrix,} \\ \text{(b) row reduction.} \end{cases}$

Ex 5.3 (More inverse matrix calculations)

Compute the inverses of the following matrices:

$$(a) \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \qquad (b) \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix} \qquad (c) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Ex 5.4 (Determining invertibility)

Determine if the following matrices are invertible or not:

$$(a) \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad (b) \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & -1 \\ 3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad (c) \begin{pmatrix} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & -4 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

Ex 5.5 (Inverting a linear transformation)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the following linear transformation:

$$T(x) = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 - 3x_3 \\ x_2 + x_3 \end{pmatrix}.$$

Prove that T is invertible and give a formula that defines the inverse transformation T^{-1} of T.

Ex 5.6 (Non-invertible matrices whose product is the identity matrix)

Find non-square matrices A and B such that $AB = I_n$ for some $n \in \mathbb{N}$ but A is not invertible.

Ex 5.7 (Invertibility of factors of an invertible matrix)

Show that if A and B are $n \times n$ matrices and AB is invertible, then A and B are also invertible.

Ex 5.8 (Invertibility of elementary matrices)

a) Let $P_{ij} \in \mathbb{R}^{m\times m}$ be an elementary matrix that interchanges the ith and the jth row of a matrix. Verify that $P_{ij}^{-1} = P_{ij}$.

b) Let $D_i(\lambda) \in \mathbb{R}^{m\times m}$ be an elementary matrix that multiplies the ith row of a matrix by a scalar $\lambda \neq 0$. Verify that $D_i(\lambda)^{-1} = D_i(\lambda^{-1})$.

c) Let $L_{ij}(\lambda)$ be an elementary matrix that adds λ times the ith row to the jth row. Verify that $L_{ij}(\lambda)^{-1} = L_{ij}(-\lambda)$.

Hint: In all cases, calculate the claimed inverse times the matrix times I_m using the effect the corresponding matrices have, e.g., compute $P_{ij}I_m =?$, then $P_{ij}P_{ij}I_m =?$.

Ex 5.9 (Multiple choice and True/False questions)

a) Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$. Then $(A^{-1})_{23}$ equals: (i) $-6/14$, (i) $-2/14$, (ii) $2/14$, (iv) $6/14$

b) Decide whether the following statements are always true or if they can be false.

(i) If A and B are invertible and of the same size, then $A + B$ is also invertible.

(ii) If $AB = AC$, then $B = C$.

(iii) If A is invertible and $AB = AC$, then $B = C$.

(iv) Every upper triangular matrix is in echelon form.

In the following, let A be a square matrix with n rows and n columns.

(v) If the equation $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$, then the solution is unique for each b.

(vi) If the columns of A are linearly independent, then they span $\mathbb{R}^n$.

(vii) If the columns of A span $\mathbb{R}^n$, then they are linearly independent.

(viii) If the columns of A are linearly independent, then its rows are also linearly independent.

(ix) If $n = 2$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ab - cd \neq 0$, then A is invertible.

(x) If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.

(xi) If there is an $n \times n$ matrix such that $AD = I$, then there is also an $n \times n$ matrix such that $CA = I$.