MATH-111(en) FALL 2025
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Homework 4

Ex 4.1 (Injective, surjective, bijective)

Each part in the below list defines a map $f : \mathbb{R}^n \to \mathbb{R}^m$. For each part, specify m and n; check whether they are linear; and check whether f is injective, surjective and/or bijective.

$$(a)\ \ f\begin{pmatrix} x \\ y \end{pmatrix} = 10x + y \qquad (b)\ \ f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + z \qquad (c)\ \ f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + 1 \\ 2x \end{pmatrix}$$

$$(d)\ \ f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + y \\ y \\ 3z + x \\ y \end{pmatrix} \qquad (e)\ \ f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + y \\ y \\ 3z + x \end{pmatrix}$$

Ex 4.2 (Surjectivity depending on a parameter)

Find all $a \in \mathbb{R}$ so that the linear transformation with matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & a & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is not surjective.

Ex 4.3 (Composition of linear maps)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^p \to \mathbb{R}^n$ be linear maps. We define their composition $f \circ g : \mathbb{R}^p \to \mathbb{R}^m$ by $(f \circ g)(x) = f(g(x))$. Show that $f \circ g$ is a linear map, too.

Ex 4.4 (Some matrix products)

Let

$$A = \begin{pmatrix} 4 & -5 & 3 \\ 5 & 7 & -2 \\ -3 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 0 & -1 \\ -1 & 5 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 5 \\ 4 & -3 \\ 1 & 0 \end{pmatrix}.$$

Compute AC, BC and CB.

Ex 4.5 (When do these matrices commute?)

Consider the matrices

$$A = \begin{pmatrix} 3 & -4 \\ -5 & 1 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 7 & 4 \\ 5 & k \end{pmatrix}.$$

For which values of k does the equality $AB = BA$ hold?

Ex 4.6 (More matrix products)

Consider the matrices:

$$A = \begin{pmatrix} 7 & 0 \\ -1 & 5 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ -4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 7 \\ -3 \end{pmatrix}, \quad D = \begin{pmatrix} 8 & 2 \end{pmatrix}.$$

If they are defined, compute the matrices

$$AB, CA, CD, DC, DBC, BDB, A^T A \text{ and } AA^T.$$

For those that are not defined, explain why.

Ex 4.7 (Multiplication by diagonal matrices)

Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

1. Compute AD and DA and explain how the rows and columns of A change when one multiplies A by D from the right and from the left.
2. Find all the diagonal matrices M of dimension 3×3 such that $AM = MA$.

Ex 4.8 (Inner and outer products)

We may consider any vector of $\mathbb{R}^n$ as an $n \times 1$ matrix. Let u and v in $\mathbb{R}^3$ be given as

$$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

1. Write u^T and v^T.

We call $u^T\ v$ the scalar product (or dot product or inner product) of the vectors u and v.

2. What is the dimension of the products $u^T\ v$ and $v^T\ u$?
3. Are these two products equal for all possible values of a, b and c? Why?

The product $u\ v^T$ is called the outer product.

4. What are the dimensions of the products $u\ v^T$ and $v\ u^T$?
5. Are these two products equal for all possible values of a, b and c? Why?

Ex 4.9 (Upper triangular matrices)

1. Compute $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for $a, b \in \mathbb{R}$.
2. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and compute the following matrices: $A^8, A^T A, AA^T$.
3. Find a matrix B such that $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
4. Prove the following statement: the transpose of an upper triangular matrix is always lower triangular.
5. Use the statement proven in the previous bullet point to argue that the transpose of a diagonal matrix is always diagonal.

Ex 4.10 (A matrix equation)
Find a solution X for the following matrix equation

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}.$$

Ex 4.11 (On invertibility of matrices)
Let $A \in \mathbb{R}^{n\times n}$. We call A invertible if there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_n$.

a) Assume that A is invertible. Show that then $Ax = 0$ has only the trivial solution $x = 0$ and that $\operatorname{span}(A_1, \dots, A_n) = \mathbb{R}^n$, where $A_1, \dots, A_n$ are the columns of A.

b) Assume that $n = 2$ and write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that A is invertible if and only if $ad - bc \neq 0$.

Hint: If $ad - bc = 0$, consider first the case $a = b = 0$ and otherwise the vector $x = \begin{pmatrix} -b \\ a \end{pmatrix}$. If $ad - bc \neq 0$, try the matrix $C = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Ex 4.12 (Multiple choice and True/False questions)

a) What x satisfies the following equation?

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 7 & -3 & -3 \\ -1 & x & 0 \\ -1 & 0 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(A) $x = -1$
(B) $x = 1$
(C) There is no such x.
(D) There are multiple such x's.

b) Decide whether the following statements are always true or if they can be false.

i) Let $A \in \mathbb{R}^{n\times n}$ for some $n \in \mathbb{N}$. Then $(A^2)^T = (A^T)^2$.
ii) Let $A \in \mathbb{R}^{n\times n}$ for some $n \in \mathbb{N}$. Then $A^T A = AA^T$.
iii) Let $A, B \in \mathbb{R}^{n\times n}$ for some $n \in \mathbb{N}$. Then $(AB)^2 = A^2B^2$.
iv) The transpose of a sum of matrices equals the sum of their transposes.
v) The transpose of a product of matrices equals the product of their transposes in the same order.