MATH-111(en) FALL 2025
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Homework 3

Ex 3.0 (Delay)
Finish Exercise 2.11 from last week if you have not done so yet.

Ex 3.1 (Examples of linear (in)dependence)
Part 1 : Are the following sets of vectors linearly independent or linearly dependent ?

(a) $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}$ (b) $\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\4\\1\end{pmatrix}$ (c) $\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

Part 2 : Are the following sets of vectors linearly independent or linearly dependent ? For this part, use Theorems 1.7, 1.8, and/or 1.9 to answer.

(a) $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}, \begin{pmatrix}10\\11\\12\end{pmatrix}$ (b) $\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\4\\1\end{pmatrix}$ (c) $\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\4\\1\end{pmatrix}$

Part 3 : Consider the following set of vectors :

$$v_1 = \begin{pmatrix}2\\4\\2\end{pmatrix}, v_2 = \begin{pmatrix}1\\2\\1\end{pmatrix}, v_3 = \begin{pmatrix}1\\3\\1\end{pmatrix}$$

First, show that they are linearly dependent. Next, show that v_3 cannot be written as a linear combination of v_1 and v_2. Finally, explain how this does not contradict Theorem 1.7.

Ex 3.2 (Proof of Corollary 1.10)
Use results from class to prove the following statements : For a matrix $A \in \mathbb{R}^{m\times n}$ with columns $A_1, \dots, A_n \in \mathbb{R}^m$, the following hold

a) for all $b \in \mathbb{R}^m$ the equation $Ax = b$ has at most one solution if and only if the columns of A are linearly independent ;
b) for all $b \in \mathbb{R}^m$ the equation $Ax = b$ has at least one solution if and only if $\operatorname{span}(A_1, \dots, A_n) = \mathbb{R}^m$;
c) for all $b \in \mathbb{R}^m$ the equation $Ax = b$ has exactly one solution if and only if $\operatorname{span}(A_1, \dots, A_n) = \mathbb{R}^m$ and $\{A_1, \dots, A_n\}$ are linearly independent.

Ex 3.3 (An equivalent definition of linear maps)
Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Show that f is linear if and only if

$$f(\lambda x + y) = \lambda f(x) + f(y) \qquad \text{for all } x, y \in \mathbb{R}^n,\ \lambda \in \mathbb{R}.$$

Hint: For one implication, check the lecture on Tuesday for a more general argument. For the other implication, insert special values of λ or y.

Ex 3.4 (The weekly linear system : matrix equations and linear (in)dependence)

For each of the following matrix equations : (i) solve the equation. (ii) From the solution to the equation, deduce whether the columns of the coefficient matrices are linearly independent or linearly dependent.

$$\text{(a)} \begin{pmatrix} 2 & -5 & 8 \\ -2 & -7 & 1 \\ 4 & 2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 1 & -3 & 7 \\ -2 & 1 & -4 \\ 1 & 2 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Ex 3.5 (Linear (in)dependence depending on a parameter)
For which values of $a \in \mathbb{R}$ are the following vectors linearly dependent ?

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ a \\ 2 \end{pmatrix}$$

Ex 3.6 (Vectors in the image of a linear transformation)
Consider the linear transformation (function) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1 - 2x_2 \\ -x_1 \\ x_1 - 2x_2 \end{pmatrix}.$$

1. Find $x \in \mathbb{R}^2$ such that $T(x) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Are there any more such vectors x ?
2. Is there an $x \in \mathbb{R}^2$ such that $T(x) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$?
3. Is there any vector $\mathbf{b}$ such that $T(x) = \mathbf{b}$ has more than one solution ?

Ex 3.7 (Visualizing linear transformations)

For the linear transformations given by the following matrices, draw a picture to show how they transform the unit square $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.

$$\text{a)} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad \text{b)} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad \text{c)} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Ex 3.8 (Representing linear transformations with matrices)
Find the matrices of the transformations T determined by the equations below.

1. $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2z - y \\ 3y - 2x \\ 4x - 3z \end{pmatrix}.$
2. $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4.$
3. $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}.$

   Hint: Express the vectors e_1, e_2, e_3 as linear combination of the vectors for which you know the image and then use linearity to compute what you need.

Ex 3.9 (Injectivity/Surjectivity of linear maps)
For each of the following matrices, determine if the corresponding linear transformation is injective and/or surjective.

$$A = \begin{pmatrix} -2 & 4 \\ 5 & 7 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

Ex 3.10 (Multiple choice and True/False questions)

a) Let

$$a_1 = \begin{pmatrix} 3 \\ -5 \\ 2 \\ 8 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ h-10 \\ 13+h \\ 10 \end{pmatrix}$$

where h is a real number. For what value of h is the vector b in the plane generated by a_1 and a_2 ?

$$(A) \quad h = -6 \qquad (B) \quad h = 0 \qquad (C) \quad h = -3 \qquad (D) \quad h = 10.$$

b) For an $m \times n$ matrix A whose columns generate $\mathbb{R}^m$ it is always true that...

(A) The equation system $Ax = \mathbf{0}$ has a non-trivial solution.

(B) The echelon form of A has a pivot in each row.

(C) There are vectors b of $\mathbb{R}^m$ for which the system of equations $Ax = b$ is not consistent.

(D) The echelon form of the A matrix has a pivot in each column.

c) Decide whether the following statements are always true or if they can be false.

(i) The columns of any 4×5 matrix are linearly dependent.

(ii) If x and y are linearly independent, and if $\{x, y, z\}$ is linearly dependent, then z is in $\text{Span}\{x, y\}$.

(iii) If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

(iv) If x and y are linearly independent, and if z is in $\text{Span}\{x, y\}$, then $\{x, y, z\}$ is linearly dependent.

(v) A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.