MATH-111(en) FALL 2025
Linear Algebra
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Homework 2

Ex 2.1 (The weekly linear system)
Solve the following linear system and write the solution in parametric form.

$$\begin{cases} x + 2y + 2t = 3 \\ y + z - 2t = 0 \\ x + 3y + z + t = 5 \\ 2x + 5y + z + 4t = 10 \end{cases}$$

Ex 2.2 (On $B(Ax) \neq A(Bx)$)
We first observe that : If $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$, then $Ax \in \mathbb{R}^m$ is well-defined. If $B \in \mathbb{R}^{p\times m}$, then the element $B(Ax) \in \mathbb{R}^p$ is well-defined.

a) Determine all values of m and p (depending on n) such that $A(Bx)$ is also defined.

b) For $n = 2$, consider the matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Show that for $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ it holds that $B(Ax) \neq A(Bx)$. Find all vectors $x \in \mathbb{R}^2$ such that $B(Ax) = A(Bx)$.

Ex 2.3 (Conversion to parametric vector form)
Write all solutions to the linear systems from Problems 1, 2, and 3 from Homework 1 in parametric vector form.

Ex 2.4 (A system with a parameter)
For

$$A = \begin{pmatrix} 1 & -1 & -3 \\ 2 & 0 & 1 \\ 0 & 2 & 7 \end{pmatrix}, x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, b = \begin{pmatrix} 3 \\ 0 \\ a \end{pmatrix},$$

find an a such that $Ax = b$ is consistent, and write the solution in this case in parametric vector form.

Ex 2.5 (Consistency for all right-hand sides ?)
Let

$$A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}.$$

Is the equation $Ax = b$ consistent for all choices of $b \in \mathbb{R}^2$? Determine the set of $b \in \mathbb{R}^2$ for which the equation $Ax = b$ is consistent.

Ex 2.6 (Homogeneous and inhomogeneous systems)

Consider the two systems

$$\begin{cases} x_1 - 3x_2 - 2x_3 &= 0 \\ x_2 - x_3 &= 0 \\ -2x_1 + 3x_2 + 7x_3 &= 0 \end{cases} \qquad \begin{cases} x_1 - 3x_2 - 2x_3 &= -5 \\ x_2 - x_3 &= 4 \\ -2x_1 + 3x_2 + 7x_3 &= -2 \end{cases}.$$

For each of these systems, write the solution in its parametric vector form and give a geometric description of the solution space. For example, say what "shape" it is and whether or not it contains the origin.

Ex 2.7 (A homogeneous equation)
For the following matrix A, write the solution of $Ax = 0$ in its parametric vector form.

$$A = \begin{pmatrix} 1 & 6 & 0 & 8 & -1 & -2 \\ 0 & 0 & 1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Ex 2.8 (On homogeneous systems)
Recall that a system $Ax = b$ is *homogeneous* if $b = 0$ (where 0 is the vector in which every entry is zero). Determine if the following statements are true or false. Justify your answer. (*Justify* means : if it is true explain/prove why ; if it is false, give a counterexample.)

1. If x is a solution to a homogeneous system, then $2x$ is also a solution.
2. If x is a solution to an inhomogeneous system, then $2x$ is also a solution.
3. If x is a solution of $Ax = 0$ and y is a solution of $Ay = b$, then $v = x + y$ is a solution of $Av = b$.
4. If x is a solution of $Ax = 0$ and y is a solution of $Ay = b$ where $b \neq 0$, then $v = 2x + 3y$ is a solution of $Av = b$.

Ex 2.9 (Two related systems)
Solve the following two systems of linear equations and write the solutions in parametric vector form. What is the connection between the solution sets of the two systems ?

$$\begin{cases} x + y - 3z &= 0 \\ 3x + 7y - 13z &= 0 \\ x - y - z &= 0 \end{cases} \qquad \begin{cases} x + y - 3z &= 0 \\ 3x + 7y - 13z &= -4 \\ x - y - z &= 2 \end{cases}$$

Ex 2.10 (Multiple choice and True/False questions)

(a) Let

$$A = \begin{pmatrix} 2 & 3 & -3 \\ -2 & 0 & 5 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}.$$

Then the solution $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of the matrix equation $Ax = b$ is such that

$(A)\quad x_1 = 3 \qquad (B)\quad x_1 = 1 \qquad (C)\quad x_1 = 4 \qquad (D)\quad x_1 = -2.$

(b) The Linear System

$$\begin{cases} x - 7y + 2z - 13t = 5 \\ 2x - 4y + 2z - 12t = 3 \\ 3x + y + 2z - t = 1 \\ 3x + 2y + 2z + 4t = 0 \end{cases}$$

has :

(A) no solution

(B) a unique solution

(C) a straight line as its solution space

(D) a plane as its solution space.

c) Decide whether the following statements are always true or if they can be false.

(i) A vector b is a linear combination of the columns of a matrix A if and only if the equation $Ax = b$ has at least one solution.

(ii) If the augmented matrix $[A\ b]$ has a pivot position in every row, then the equation $Ax = b$ is inconsistent.

(iii) The solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the equation $Ax = 0$.

(iv) The equation $Ax = b$ is homogeneous if the zero vector is a solution.

Ex 2.11 (Linear combinations and linear dependency : Proofs)

(a.) Finish the proof of Theorem 1.7 from class.

(b.) Prove Theorem 1.8 from class.

(c.) Prove Theorem 1.9 from class.

Some advice on proof writing : if you cannot come up with a proof from the top of your head : try to first state the assumption then the goal and write both out according to the definition. (See e.g. first part of proof of Theorem 1.7 from class.) Then try to work from one towards the other. If you still cannot do it, it may be fruitful to make up an example (choose a couple of vectors in $\mathbb{R}^3$ with actual numeric entries) that illustrates the given scenario. This might give you an idea of where to start the formal proof. Coming up with your own examples to explain something to yourself is a crucial part of becoming independent in your mathematical thinking process !