MATH-111(en) FALL 2025
Linear Algebra
developed by Annina Iseli
taught by Andrei Negut

Homework 1

Ex 1.1 (Solving linear systems I)
Solve the following systems of linear equations by finding the augmented matrix and using row elimination. Once you solved a system, make sure to display your solution in correct notation.

$$a)\ \begin{cases} x+y &= 5 \\ 2x-5y &= 4 \end{cases} \qquad b)\ \begin{cases} x+y+z &= 3 \\ x-y &= 0 \\ x &= -46 \end{cases} \qquad c)\ \begin{cases} 2x+3y+z &= 0 \\ x-y+z &= 1 \\ 3x+2y+2z &= 1 \end{cases}$$

Ex 1.2 (Solving linear systems II)
Find the augmented matrix of the following linear systems and use it to solve them.

$$a)\ \begin{cases} w-x+z &= 1 \\ x-2y-z &= 0 \\ w+3y &= 2 \end{cases} \qquad b)\ \begin{cases} 2x-5y+4z &= 0 \\ x+y+z &= 0 \\ 4x-3y+6z &= 1 \end{cases} \qquad c)\ \begin{cases} x-2y+3z &= 1 \\ 2x-4y+6z &= 2 \\ -x+2y-3z &= -1 \end{cases}$$

Ex 1.3 (Solving linear systems III)
Find the augmented matrix of the following linear systems and use it to solve them.

$$a)\ \begin{cases} w-x+y-z &= 1 \\ w+z &= 2 \end{cases} \qquad b)\ \begin{cases} x+y &= 0 \\ 3x+5y &= 2 \\ 2x+4y &= 2 \end{cases}$$

Ex 1.4 (Linear systems with a parameter)
Determine for which $a \in \mathbb{R}$ the following system has no solution, a unique solution, or infinitely many solutions.

$$\begin{cases} x-2y+3z &= 2 \\ x+3y-2z &= 5 \\ 2x-y+az &= 1 \end{cases}$$

Ex 1.5 (Solvability of parameter-dependent systems)
Determine the values of h for which the following matrices are the augmented matrices of a consistent linear system. (A linear system is called *consistent* if it has at least one solution. It is *inconsistent* if there exists no solution.)

$$\text{(a)} \begin{pmatrix} 1 & -3 & h \\ -2 & 6 & -5 \end{pmatrix}, \qquad \text{(b)} \begin{pmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{pmatrix}.$$

Ex 1.6 (Linear combinations)

a) For the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 11 \\ 16 \\ 21 \end{pmatrix}$, find $l, m \in \mathbb{R}$ such that $\mathbf{c} = l\mathbf{a} + m\mathbf{b}$.

b) Find all $a \in \mathbb{R}$ such that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ 2 \end{pmatrix} \right\}$.

Ex 1.7 (Linearity of linear systems)
Let

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n & = b_1 \\ \vdots & \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n & = b_m \end{cases}$$

be a general linear system.

a) Show that if $(s_1, \ldots, s_n)$ is a solution for the right-hand side $(b_1, \ldots, b_m)$ and $\lambda \in \mathbb{R}$ is a real number, then $(\lambda s_1, \ldots, \lambda s_n)$ is a solution for the right-hand side $(\lambda b_1, \ldots, \lambda b_m)$.

b) Show that if $(s_1, \ldots, s_n)$ is a solution for the right-hand side $(b_1, \ldots, b_m)$ and $(t_1, \ldots, t_n)$ is a solution for the right-hand side $(c_1, \ldots, c_m)$, then $(s_1 + t_1, \ldots, s_n + t_n)$ is a solution for the right-hand side $(b_1 + c_1, \ldots, b_m + c_m)$.

Remark: i) From a formal perspective, these two properties are the definition of a linear system.
ii) This exercise requires an rigorous proof.

Ex 1.8 (True/False with justification)
Determine if the following statements are true or false. Justify your answer, either with a proof or a counterexample.

1. If $u \in \operatorname{Span}\{v_1, \ldots, v_n\}$, then $v_1 \in \operatorname{Span}\{u, v_2, \ldots, v_n\}$.
2. For $u, v \in \mathbb{R}^3$ with $u \neq v$, $\operatorname{Span}\{u, v\}$ is always a plane.

Ex 1.9 (Multiple choice and True/False questions)

a) Multiple Choice : Let R be the reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 3 & 1 & 2 & -2 \\ 2 & 3 & -1 & -3 \end{pmatrix}$$

and denote by r_{ij} its entry located on the row i and column j. Then

$(A)\quad r_{13} = -2 \qquad (B)\quad r_{13} = -1 \qquad (C)\quad r_{13} = 0 \qquad (D)\quad r_{13} = 1.$

b) Multiple Choice : For each of the following statements, decide whether it always must be true or if it can be false.

(i) If the augmented matrix of a linear system has a row of the form $\begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}$, then there exists no solution.
(ii) When a linear system has free variables, then there exist infinitely many solutions.
(iii) If $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$, then $0 \in \operatorname{Span}\{v_1, \ldots, v_k\}$, where $0 \in \mathbb{R}^n$.
(iv) A linear system with right-hand side $b_1 = \ldots = b_m = 0$ always has a solution.

c) True/False : Every matrix can be brought into echelon form and that echelon form is unique.

d) True/False : Every system of linear equations with more unknowns than equations must have : (i) no solutions (ii) infinitely many solutions (iii) both are possible.