MATH-111(en) Fall 2025
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Homework 14

Ex 14.0 (Evaluation)

Please fill out the course evaluation form on Moodle, it means a lot to me!

Ex 14.1 (Orthogonal diagonalization)

Orthogonally diagonalize the matrices $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -3 & 0 & 0 \\ -3 & 12 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -3 & 12 \end{pmatrix}$

Ex 14.2 (Orthogonal diagonalization with some help)

Consider

$$A = \begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

1. Check that v_1 and v_2 are eigenvectors of A.
2. Orthogonally diagonalize the matrix A. (*Hint*: Make use of the fact that you already know two eigenvectors instead of just using the standard recipe for orthogonal diagonalization!)

Ex 14.3 (Computing an SVD)

Find a singular value decomposition for each of the following matrices:

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Ex 14.4 (SVD with higher geometric multiplicity)

Find a singular value decomposition of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Ex 14.5 (A proof using SVD)

We say that two matrices $A, B \in \mathbb{R}^{n\times n}$ are called *orthogonally similar* if there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $A = QBQ^T$. Let $A \in \mathbb{R}^{n\times n}$. Show that A^TA and AA^T are orthogonally similar.

Hint: Use a SVD of A and that the product of orthogonal matrices is orthogonal.

You do not need to memorize the definition of *orthogonally similar* for the exam. This exercise is just for you to train your proof-writing within the topics of Section 7.

Ex 14.6 (Computing SVD from eigenvectors)
Let $A \in \mathbb{R}^{2\times 4}$, $w_1, w_2 \in \mathbb{R}^4$ be such that w_1, w_2 are eigenvectors of $A^T A$, and

$$w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad Aw_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad Aw_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Find matrices U, Σ and V such that A has singular value decomposition of the form

$$A = U\Sigma V^T.$$

Hint: *You do not have to explicitly compute the matrix A in order to solve this problem! First determine the size of A and hence the sizes of U,*
Sigma and V. Then recall the recipe for computing SVD. Observe that some of the usually necessary computations can be skipped as you were already provided with the result.)

Ex 14.7 (Calculating $\exp(tA)$ and solving ODEs)
Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. (a) Compute $\exp(tA)$. (**Hint**: Diagonalize A.)

b) Solve the differential equation $x'(t) = A \cdot x(t)$ for each of the initial values:
(i) $x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, (ii) $x(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Ex 14.8 (Solving ODEs)
Solve the following system of differential equations :

$$\begin{cases} x_1'(t) &= 5x_1(t) - 4x_2(t) - 2x_3(t) \\ x_2'(t) &= -4x_1(t) + 5x_2(t) + 2x_3(t) \\ x_3'(t) &= -2x_1(t) + 2x_2(t) + 2x_3(t) \end{cases}$$

for the initial values $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 1$

Hint: *transfer it into a suitable matrix form. Then before your start investing loads of time into computations, ask yourself whether the matrix looks familiar to you.*

Ex 14.9 (A higher order ODE)
Consider the following differential equation: $y'''(t) + 4y''(t) - 4y'(t) = 0$

(a) Transform this ODE of order n into a system of ODEs of order 1 and write it in matrix-vector-form.

(b) Compute $\exp(tA)$ for $t \in \mathbb{R}$.

(c) Using the method of matrix exponentials, compute a solution $y(t)$ for the differential equation for the initial values: $y''(0) = y'(0) = 0$ and $y(0) = 1$?

Ex 14.10 ($\exp(tA)$ for a non-diagonalizable matrix)

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. a) Show that A is not diagonalizable.

b) Show by induction that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

c)* *Non-mandatory exercise.* Show that $\exp(tA) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$.

You may use the formula $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ex 14.11 (Diagonalization of a matrix exponential)
Let A be the matrix from Exercise 11.2 (see Homework 11). Diagonalize $\exp(tA)$ for $t \in \mathbb{R}$.

Ex 14.12 (Multiple choice and True/False questions)

a) Consider the matrices $A = \begin{pmatrix} -7/4 & 1/2 \\ 1/2 & 3/5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

Which among the following statements are true?

(A) A and B are orthogonally diagonalizable

(B) A is orthogonally diagonalizable and B is diagonalizable

(C) A is diagonalizable but B is not.

(D) neither A nor B are orthogonally diagonalizable.

b) Decide whether the following statements are always true or if they can be false.

(i) If $A = A^T$ and $Ax = 0$ and $Ay = y$, then $x \cdot y = 0$.

(ii) If $A = A^T$, then A has n distinct real eigenvalues.

(iii) An orthogonal matrix is orthogonally diagonalizable.

(iv) If $A \in \mathbb{R}^{m \times n}$ and if $P \in \mathbb{R}^{m \times m}$ is orthogonal, then A and PA have the same singular values.

(v) If $A \in \mathbb{R}^{n \times n}$, then A and $A^T A$ have the same singular values.

(vi) If A is orthogonally diagonalizable, then $\exp(tA)$ is orthogonally diagonalizable.