MATH-111(en) FALL 2025
Linear Algebra
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Homework 13

Ex 13.1 (Using the Gram–Schmidt process)
Let W be the subspace of $\mathbb{R}^4$ spanned by the basis vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \\ -1 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix}.$$

a) Construct an orthogonal basis for W using the Gram–Schmidt process.

b) Consider $A = [v_1 \quad v_2 \quad v_3]$ having the vectors v_1, v_2, v_3 as columns. Find out a QR decomposition of A.

Ex 13.2 (Finding an orthonormal basis)
Find an orthonormal basis for the span of the following vectors: $\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$

Ex 13.3 (QR factorization)
Find a QR factorization for each of the following matrices:

$$A = \begin{pmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{pmatrix}$$

Ex 13.4 Proof that for any matrix $A \in \mathbb{R}^{m \times n}$, the following statements are equivalent:

(i) For every $b \in \mathbb{R}^m$, the equation $Ax = b$ has a unique least square solution.

(ii) $A^T A$ is invertible

(iii) The columns of A are linearly independent.

Ex 13.5 (A least-squares problem)
Find all least-squares solution $x^\star$ of the system $Ax = b$ and their least square errors $\|Ax^\star - b\|$.

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

Ex 13.6 (Another least-squares problem)
Find all least-squares solution $x^\star$ of the system $Ax = b$ and their least square errors $\|Ax^\star - b\|$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

Ex 13.7 (QR decomposition for a least-square problem) Consider

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}.$$

a) Show that

$$A = \begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}.$$

b) Use this QR decomposition of A to find the least squares solution to the equation $Ax = b$.

Ex 13.8 (Linear regression)

(a) Find the straight line that best approximates (in the sense of least squares) the following data points in $\mathbb{R}^2$: $(2,1), \quad (5,2), \quad (7,3), \quad (8,3)$

(b) Draw a picture that illustrates the data points and the line that best approximates them.

Ex 13.9 (Linear regression)
Assume that you measure the measure the temperature near a chemical experiment at times $t = 1, 2, 3, 4, 5, 6$. The measurements y (ordered by time) that you obtain are $20, 30, 35, 40, 45, 45$. Find a affine function $f(t) = y$ approximating your data with minimal least square error. Also, give the value of the least square error.

Ex 13.10 (Repetition of old topics)

(a) Prove that the set of symmetric matrices in $\mathbb{R}^{n\times n}$ are a subspaces of $\mathbb{R}^{n\times n}$.

(b) Prove that the dimension of this subspaces is $\frac{n(n+1)}{2}$

(c) What is the dimension of the space of anti-symmetric matrices?

Ex 13.11 (Two quick proofs)

a) Let $A \in \mathbb{R}^{n\times n}$. Show that $A^T = A$ if and only if $Ax \cdot y = x \cdot Ay$ for all $x, y \in \mathbb{R}^n$.

b) Let $Q, U \in \mathbb{R}^{n\times n}$ be orthogonal matrices. Show that QU and Q^{-1} are also orthogonal.

Ex 13.12 (Multiple choice and True/False questions)

a) Let the matrix $A = \begin{pmatrix} -3 & -2 \\ 0 & 1 \\ 2 & -3 \end{pmatrix}$ and the vector $b = \begin{pmatrix} -6 \\ 11 \\ 17 \end{pmatrix}$.

Then the solution in the sense of the least squares $\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}$ of the equation $Ax = b$ is such that

$(A)\quad \hat{x}_2 = -2 \qquad (B)\quad \hat{x}_2 = 3 \qquad (C)\quad \hat{x}_2 = -1 \qquad (D)\quad \hat{x}_2 = 1$

b) Decide whether the following statements are always true or if they can be false.

(i) Let $y \in \mathbb{R}^n$ and W be a subspace of $\mathbb{R}^n$. Then $y - \text{proj}_W(y)$ is orthogonal to W.

(ii) If W is a subspace of $\mathbb{R}^n$, then $\text{proj}_W \circ \text{proj}_W = \text{proj}_W$, where $\circ$ denotes the composition of maps.

(iii) If $A = QR$ and Q has orthonormal columns, then $R = Q^T A$.

(iv) A least-squares solution of $Ax = b$ is a vector $x_0 \in \mathbb{R}^n$ such that $Ax_0 = \text{proj}_{\text{Col}(A)}(b)$.

(v) If $b \in \text{Col}(A)$, then the least-squares solutions are exactly the solution of the equation $Ax = b$.

(vi) The line of regression is unique provided we have measurements for at least two different inputs.