MATH-111(en) FALL 2025
Linear Algebra

developed by Annina Iseli
taught by Andrei Negut

Homework 10

Ex 10.1 (Calculus with complex numbers)

a) Find the real and imaginary part of $z = \dfrac{1}{1+i} - \dfrac{2+3i}{2+i}$.

b) Calculate the modulus of the following complex numbers:

$$z_1 = i, \quad z_2 = 2 - 2i, \quad z_3 = -4i.$$

c) Given $z \in \mathbb{C}$ we define $w \in \mathbb{C}$ by

$$w = \varepsilon \cdot \sqrt{\frac{|z| + \mathrm{Re}(z)}{2}} + i \cdot \sqrt{\frac{|z| - \mathrm{Re}(z)}{2}},$$

where $\varepsilon = \pm 1$ is the sign of $\mathrm{Im}(z)$ (convention: the sign of zero equals 1). Show that $w^2 = z$, i.e., w is a square-root of z. Use this result to find a square-root of i.

Ex 10.2 (The matrix of a linear map relative to two bases)

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 4x_1 + 3x_2 \\ 10x_1 - 8x_2 \\ x_1 + 2x_2 \end{pmatrix}.$$

Consider the bases $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$ and $\mathcal{C} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$. Give the matrix $M = \underset{\mathcal{C} \leftarrow \mathcal{B}}{[T]}$ that represents T going from $\mathcal{B}$ to $\mathcal{C}$; in other words, the matrix M such that $[T(\mathbf{v})]_{\mathcal{C}} = M \cdot [\mathbf{v}]_{\mathcal{B}}$.

Ex 10.3 (Finding a basis for an eigenspace)

Find a basis for the eigenspace of the eigenvalue $\lambda = 3$ of the matrix

$$B = \begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix}.$$

Ex 10.4 (Some eigenvalues and eigenvectors)

Consider the matrices

$$C = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad D = \begin{pmatrix} 5 & 4 \\ -1 & 1 \end{pmatrix}.$$

Find the eigenvalues of both, and give an eigenvector for each eigenvalue.

Ex 10.5 (More eigenvalues and eigenvectors)
Find the eigenvalues of

$$E = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and give an eigenvector for each.

Ex 10.6 (Eigenvalues of AB and BA)
Let A and B be two $n \times n$ matrices. Show that AB and BA have the same eigenvalues.
Hint: Distinguish the cases $\lambda = 0$ and $\lambda \neq 0$.

Ex 10.7 (Some statements about eigenvalues and eigenvectors)

1. Show that if λ is an eigenvalue of an invertible matrix A then λ^{-1} is an eigenvalue of A^{-1}.
   Hint: use a non zero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$.
2. Show that A and A^T have same eigenvalues.
   Hint: Find a relation between $\det(A - \lambda I)$ and $\det(A^T - \lambda I)$.
3. Let A be an $n \times n$ matrix such that the sum of each row's elements is equal to the same number r.

   (i) Show that r is an eigenvalue of A.
   Hint: find an eigenvector.

   (ii) *[This is a harder and therefore not exam-relevant but still fun problem:]*
   Show that if all the elements in A are in addition positive then the absolute value of any other eigenvalue is less than r.
   Hint: You will need to use the triangle inequality.
4. Let A be an $n \times n$ matrix such that the sum of each column's elements is equal to the same number c. Show that c is an eigenvalue of A.
   Hint: Use the results from the previous parts.

Ex 10.8 (Even more eigenvalues and eigenvectors)
Consider the matrices

$$A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}.$$

For each of the matrices A and B: find its characteristic polynomial, its eigenvalues as well as their eigenvectors.

Hint: The characteristic polynomial of B is of degree 3 with no simple structure. To guess a root, you can use the rational root theorem: for a polynomial with **rational coefficients** $a_0 \ldots, a_n \in \mathbb{Q}$ any rational root (if it exists) has to be of the form a/b, where a divides the constant coefficient a_0 and b divides the leading coefficient a_n.

Ex 10.9 (Who comes up with these titles?)
Let A and B be the following matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

For each matrix find out the characteristic polynomials, eigenvalues and the corresponding eigenvectors in $\mathbb{R}$.

Ex 10.10 (Rotation matrices 2×2)

Let α be an angle in $[0, 2\pi)$ and define the matrix R_α by:

$$R_\alpha := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

(a) Verify computationally that the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ induced by R_α (i.e. $f(v) = R_\alpha v$ for all $v \in \mathbb{R}^2$) is the counter-clockwise rotation around the origin by angle α.

Hint: Write $v = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ for some angel $\theta \in [0, 2\pi)$ (i.e. express v in polar coordinates), then compute $R_\alpha v$ and simplify using angle sum theorems of trigonometric functions. Then explain (e.g. in words and a drawing) why your computations show that f is the desired rotation.

(b) Compute all eigenvalues of R_α.

(c) For each eigenvalue, compute one eigenvector of R_α.

Ex 10.11 (Multiple choice and True/False questions)

a) (i) Let the matrix

$$A = \begin{bmatrix} -4 & 2 & -2 \\ 4 & -6 & \alpha \\ 5 & \beta & 7 \end{bmatrix}$$

with the parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

So, for all $\alpha \in \mathbb{R}$, the number -2 is an eigenvalue of the matrix A if

(A) $\beta = 4$ (B) $\beta = -5$ (C) $\beta = -3$ (D) $\beta = -2$

(ii) Let the matrix

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 5 & \alpha & 2 \\ \beta & 1 & 4 \end{bmatrix},$$

with the parameters $\alpha \in \mathbb{R}$ et $\beta \in \mathbb{R}$.

So, for all $\alpha \in \mathbb{R}$, the number 1 is an eigenvalue of the matrix A if

(A) $\beta = -3$ (B) $\beta = 5/3$ (C) $\beta = -2$ (D) $\beta = 3$

b) Decide whether the following statements are always true or if they can be false.

(i) If v_1 and v_2 are linearly independent eigenvectors of a matrix A, then they correspond to different eigenvalues.

(ii) The sum of two eigenvectors of a matrix is again an eigenvector.

(iii) If A is invertible and v is an eigenvector of A, then v is also an eigenvector of A^{-1}.

(iv) The eigenvalues of a matrix are the diagonal elements.

(v) Elementary row operations do not change the eigenvalues of a matrix.

(vi) A square matrix is invertible if and only if 0 is not an eigenvalue of A.