Inner product spaces

- $K = \mathbb{R}$ or $\mathbb{C}$, V, W are K-vector spaces

<u>Definition</u> An inner product on V is a function $\langle \cdot , \cdot \rangle : V \times V \longrightarrow K$ satisfying that

- $\langle v, v \rangle \in \mathbb{R}_{\geq 0}$ $\forall v \in V$ & $\langle v, v \rangle = 0 \Leftrightarrow v = 0_V$
- $\langle \cdot , v \rangle : V \longrightarrow K$ is linear $\forall v \in V$

  $\hookrightarrow$ so, an element of V^*
  if $\dim(V) < \infty$ & V has an inner product, any linear functional on V will be of this form (Riesz rep. thm.)
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$ $\forall u, v \in V$

! it will not be linear on the 2nd entry
$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ but $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$

* If $K = \mathbb{R}$ this just means that $\langle \cdot , \cdot \rangle$ is a symmetric bilinear form which is positive definite.

$\rightsquigarrow$ If $\dim(V) = n$, $\exists A \in \mathbb{M}_{n \times n}(\mathbb{R})$ symmetric & <u>diagonalisable</u> with only > 0 eigenvalues

automatic A is self-adjoint (spectral thm)

Examples:

1) "Dot product" on $\mathbb{R}^n$:

$$\underset{x}{\underset{\shortparallel}{(x_1, \dots, x_n)}} \cdot \underset{y}{\underset{\shortparallel}{(y_1, \dots, y_n)}} = \overset{\langle x, y \rangle}{\overset{\shortparallel}{\sum_{i=1}^{n} x_i y_i}}$$

2) $V = \mathcal{C}([0,1], \mathbb{R})$

$$\langle f, g \rangle := \int_0^1 f(t) g(t) \, dt$$

3) $V = \mathbb{C}^2$, $K = \mathbb{C}$ $\quad \alpha_1, \alpha_2 \in \mathbb{R}_{>0}$

$$\langle (z_1, z_2), (w_1, w_2) \rangle = \alpha_1 z_1 \overline{w_1} + \alpha_2 z_2 \overline{w_2}$$

Once we have an inner product on V, we can talk about distances!

<u>Def</u> $(V, \langle \cdot, \cdot \rangle)$ $\quad \|v\| := \langle v, v \rangle^{1/2} \in \mathbb{R}_{\geq 0}$!

properties of a norm	induced distance function (metric)
$\|v\| \geq 0 \ \forall v$	
$\|v\| = 0 \iff v = 0$	$d(u, v) := \|u - v\|$
$\|\lambda \cdot v\| = \lvert\lambda\rvert \cdot \|v\| \quad \forall \lambda \in K, \forall v \in V$	
$\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in V$	

$\rightsquigarrow$ we can also talk about "angles" & orthogonality:

<u>Def</u> $u, v \in V$ are orthogonal $\Leftrightarrow$ $\langle u, v \rangle = 0$

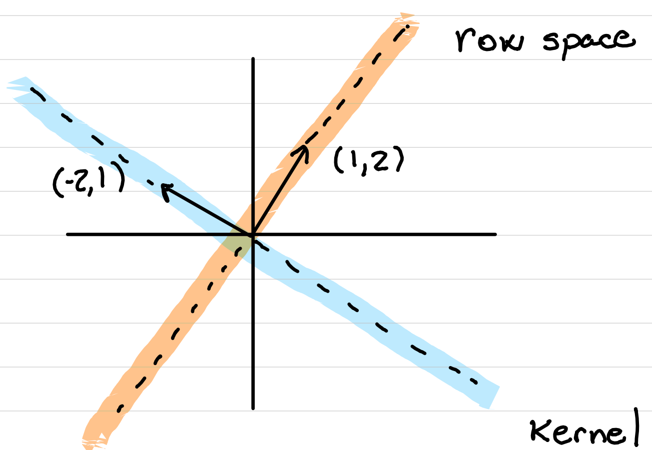
<u>Example</u> Consider $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ &

$$\varphi_A : \mathbb{M}_{n\times 1}(\mathbb{R}) \longrightarrow \mathbb{M}_{m\times 1}(\mathbb{R})$$
$$X \longmapsto A \cdot X$$

$\Rightarrow$ $X \in \operatorname{Ker}(\varphi_A) \Leftrightarrow X$ is orthogonal to each <u>row</u> of A

(w.r.t. the standard Euclidean product)

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

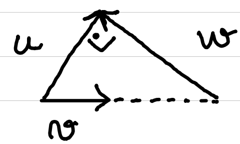


$\rightsquigarrow$ dim(row space) + dim(Ker) = dim(domain)

$\rightsquigarrow$ this gives a way of expressing when A is invertible

In particular, we can consider

- orthogonal complements & orthogonal decompositions



$u = \alpha v + w$, $u \perp w$

$\alpha = \dfrac{\langle u, v \rangle}{\|v\|^2}$

- orthogonal / orthonormal bases (Gram-Schmidt)
- operators which are "compatible" with the inner product
  - ↳ isometries, interesting subgroups of $GL(n, K)$ ...
  - ↳ self-adjoint, normal ...

<u>Q</u> When does an operator on an inner product space admit a diagonal matrix w.r.t some orthonormal basis? Same for upper triangular.

⇝ (Spectral thm)

e.g. $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \underset{\text{similar to}}{\rightsquigarrow} \begin{pmatrix} 2 & & \\ & 1 & \\ & & 0 \end{pmatrix}$

$v_1 = (-1/\sqrt{2}, 0, 1/\sqrt{2})$, $v_2 = (0, 1, 0)$, $v_3 = (1/\sqrt{2}, 0, 1/\sqrt{2})$

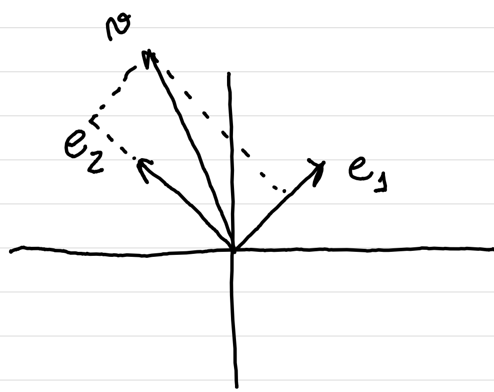
Why finding such types of basis is desirable?

- Suppose $B = (e_1, \dots, e_n)$ is orthonormal
  $\left(\Leftrightarrow \langle e_i, e_j \rangle = \delta_{ij} \right)$

then $v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n \quad \forall v \in V$

e.g. In $\mathbb{R}^2$: $e_1 = \left(1/\sqrt{2}, 1/\sqrt{2} \right)$

$e_2 = \left(-1/\sqrt{2}, 1/\sqrt{2} \right)$



$v = (-1, 4)$

$$v = \frac{3}{\sqrt{2}} e_1 + \frac{5}{\sqrt{2}} e_2 = \left(\frac{3}{2}, \frac{3}{2} \right) + \left(\frac{-5}{2}, \frac{5}{2} \right)$$

Solutions to $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \right.$