Analysis I (English)
Roberto Svaldi and Stefano Filipazzi
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Analysis 1 - Exercise Set 13

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. State on which closed intervals the following functions are integrable and compute the antiderivatives.

   (a) $f(x) = e^x$;

   (b) $f(x) = \sinh(x)$;

   (c) $f(x) = (ax+b)^s$ with $s \in \mathbb{Z}$ and $a, b \in \mathbb{R} \setminus \{0\}$;

   (d) $f(x) = \cos(x)^3$;

   (e) $f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0; \end{cases}$

   (f) $f(x) = \cot(x)$, $\cot(x) := \frac{\cos(x)}{\sin(x)}$.

   (g) $f(x) = |x|^s$, $s > 0$.

Solution:

(a) The function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval. The antiderivative is $F(x) = e^x + c$.

(b) The function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval. We compute the antiderivative formally as follows:

$$F(x) = \int \frac{e^x - e^{-x}}{2} dx = \frac{e^x + e^{-x}}{2} + c = \cosh(x) + c$$

where c is the constant of integration.

(c) If $s \geq 0$ the function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval. If $s \leq -1$ the function is continuous and hence integrable on every closed interval contained in $\mathbb{R} \setminus \{-\frac{b}{a}\}$. We compute the antiderivative formally as follows: if $s \neq -1$ then

$$F(x) = \int (ax+b)^s dx = \frac{1}{a} \int a(ax+b)^s dx = \frac{1}{a(s+1)}(ax+b)^{s+1} + c.$$

If $s = -1$, we make the substitution $u = ax + b$ so $x = \frac{1}{a}(u - b)$ with derivative $\frac{1}{a}$

$$\int (ax+b)^{-1} dx = \int \frac{u^{-1}}{a} du = \frac{1}{a} \log|u| + c = \frac{1}{a} \log|ax+b| + c.$$

(d) The function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval.

We have two possible strategies. The first one consists in writing $\cos^3(x) = \cos(x)(1-\sin^2(x))$, so that the integral becomes

$$\int cos(x)\,dx - \int \sin^2(x)\cos(x)\,dx,$$

where the first integral can be computed directly, while for the second one we set $u(x) = \sin(x)$ and $u'(x) = \cos(x)$. Then, we get

$$\begin{aligned}\int cos(x)\,dx - \int \sin^2(x)\cos(x)\,dx &= \sin(x) - \int u^2(x)u'(x)\,dx \\ &= \sin(x) - \frac{1}{3}u^3(x) + c \\ &= \sin(x) - \frac{\sin^3(x)}{3} + c.\end{aligned}$$

Alternatively, we use integration by parts with $g(x) = \cos(x)^2$ and $h(x) = \sin(x)$, so that $g'(x) = -2\cos(x)\sin(x)$ and $h'(x) = \cos(x)$

$$\int \cos(x)^3 dx = \int gh' dx = gh - \int g'h dx = \cos(x)^2\sin(x) + 2\int \cos(x)\sin(x)^2 dx$$

$$= \cos(x)^2\sin(x) + 2\int \cos(x)(1-\cos(x)^2)dx = \cos(x)^2\sin(x) + 2\sin(x) - 2\int \cos(x)^3 dx$$

We use the equation

$$\int \cos(x)^3 dx = \cos(x)^2\sin(x) + 2\sin(x) - 2\int \cos(x)^3 dx$$

to solve for $\int \cos(x)^3 dx$, and we get

$$F(x) = \frac{1}{3}(\sin(x)(2+\cos^2(x))) + c.$$

Notice that the two answers are the same, we just need to rewrite $\cos^2(x) = 1 - \sin^2(x)$ to go from the second answer to the first answer.

(e) Since f is continuous everywhere except at $x = 0$, f is clearly integrable on any closed interval not containing 0. We claim that f is also integrable on intervals containing 0. Let $I = [a, b]$ be such interval, and let σ_n be the even partition of I into n subintervals. Notice that 0 belongs to at least one of these sub-intervals, and to at most two of these sub-intervals (this only happens if 0 is the right-end point of a sub-interval that is not the last sub-interval, so that it is also the left-end point of the subsequent one). Notice that, on every sub-interval, the infimum of f is 0, as the function is identically 0 except at 0, where it is strictly positive. To conclude, we need to show that the upper sums converge to 0. If I_i is a sub-interval and $0 \notin I_i$, then the supremum of f on I_i is 0. So, we only need to consider the sub-intervals containing 0. There, the supremum is 1, as $f(0) = 1$. Then, each such sub-interval would contribute with $1 \cdot \frac{b-a}{n}$. Since there are at most 2 such sub-intervals, the upper sum is bounded above by $\frac{2(b-a)}{n}$. As b and a are fixed, this quantity converges to 0 as $n \to +\infty$. So, f is integrable on every closed interval.

There is no antiderivative because if there was it would be $g(x) = \int_0^x f(t)dt + C = C$ and $g'(x) = 0 \quad \forall x \in \mathbb{R}$. Hence, $g'(0) \neq f(0)$.

(f) One can quickly verify that $g(x) = \ln|\sin(x)| + C$ is an antiderivative. This can be seen from $\cot(x) = \frac{(\sin(x))'}{\sin(x)}$. $\cot(x)$ is also continuous on the intervals $S_i = (k\pi, (k+1)\pi)$ and therefore integrable in any interval contained in any of the sets S_i. We will show that it is not integrable at any interval containing the endpoints of S_i. By symmetry it suffices to consider $[-\frac{\pi}{4}, \frac{\pi}{4}]$. On $[-\frac{\pi}{4}, \frac{\pi}{4}]$ we have $\cos(x) > \frac{1}{2}$. We also know $|\sin(x)| \leq x$.

Let $\Pi = \{\pi_1, \cdots, \pi_n\}$ be a partition of $[-\frac{\pi}{4}, \frac{\pi}{4}]$. There will be at least one of the intervals in Π containing 0. Call it π_i. Then $\sup_{x\in\pi_i} \cot(x) = +\infty$ and $\inf_{x\in\pi_i} = -\infty$ which shows that the Darboux sums do not converge. Hence, $\cot(x)$ is not Riemann integrable in an interval containing a discontinuity of $\cot(x)$.

(g) Since $s > 0$, the function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval. Moreover, we know that the derivative of the function $g\colon \mathbb{R}_+^* \to \mathbb{R}$, $g(x) := x^t$, $t \in \mathbb{R}$ is $g'(t) = tx^{t-1}$. Hence, an antiderivative of f is given by $G(x) = \frac{\mathrm{sgn(x)}}{s+1}|x|^{s+1}$.

2. Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 + 3x + 2$ on the interval $[1, 4]$.

Solution: First let's notice that the function is a polynomial and so is continuous on the given interval. This means that we can use the Mean Value Theorem. We have

$$\begin{aligned}
&\int_1^4 x^2 + 3x + 2\ dx = (c^2 + 3c + 2)(4-1) \\
&\Rightarrow (\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x)\,|_1^4 = 3(c^2 + 3c + 2) \\
&\Rightarrow \frac{99}{2} = 3c^2 + 9c + 6 \\
&\Rightarrow 0 = 3c^2 + 9c - \frac{87}{2}
\end{aligned}$$

This equation has the two solutions $c_1 = (-3 + \sqrt{67})/2$ and $c_2 = (-3 - \sqrt{67})/2$. Clearly the second number is not in the interval $[1, 4]$ so c_1 is the acceptable value. Note that it is possible for both numbers to be in the interval so don't expect only one to be in the interval.

3. Let

$$f(x) = \begin{cases} \sin(x) & 0 \leq x \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \leq x \leq 3 \end{cases}$$

Compute $\int_0^3 f(x)dx$.

Solution:

$$\int_0^3 f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^3 f(x)dx = \int_0^{\frac{\pi}{2}} \sin(x)dx + \int_{\frac{\pi}{2}}^3 1dx = 1 + \left(3 - \frac{\pi}{2}\right).$$

4. **True/False:** If the statement is true you should prove it. If it is false you should give a counterexample. Let F be an anti-derivative of f on $[a, b]$.

   (a) If $f(x) \leq 0$ for all $x \in [a, b]$, then $F(x) \leq 0$ for all $x \in [a, b]$.

   (b) For all $x \in [a, b]$, we have $F(x) = \displaystyle\int_a^x f(t)\,dt$.

Solution:

(a) False. Take for example $f(x) = x$ on the interval $[-2, -1]$. Then $f(x) \leq 0$ on $[-2, -1]$ but $F(x) = \frac{1}{2}x^2 > 0$ For all $x \in [-2, -1]$.

(b) False. Consider for example the constant function $f(x) = 1$ on the interval $[0, 1]$. Then $F(x) = x + 1$ is an anti-derivative of f but

$$\int_0^x f(t)\,dt = \int_0^x dt = x - 0 = x \neq x + 1 = F(x).$$

Indeed, remember that, once we know there is one anti-derivative, then there are infinitely many (we may add any constant!).

5. Show that:

   (a) if $f : [-a, a] \to \mathbb{R}$ is an integrable odd function then $\int_{-a}^{a} f(x)dx = 0$;

   (b) if $f : [-a, a] \to \mathbb{R}$ is an integrable even function then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

Solution:

(a) Using Algebra of Integrals we see

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_0^a f(x)dx + \int_{-a}^{0} f(x)dx \\ &= \int_0^a f(x)dx - \int_a^0 f(-t)dt \\ &= \int_0^a f(x)dx + \int_a^0 f(t)dt \\ &= \int_0^a f(x)dx - \int_0^a f(t)dt = 0,\end{aligned}$$

where in the second line we make the change of variable $t = -x$ and $dt = -dx$, in the third line we use that f is odd to get $f(-t) = -f(t)$, and in the last step we flip sign by swapping the extrema of integration.

(b) Using Algebra of Integrals we see

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_0^a f(x)dx + \int_{-a}^{0} f(x)dx \\ &= \int_0^a f(x)dx - \int_a^0 f(-t)dt \\ &= \int_0^a f(x)dx - \int_a^0 f(t)dt \\ &= \int_0^a f(x)dx + \int_0^a f(t)dt = 2\int_0^a f(x)dx,\end{aligned}$$

where in the second line we make the change of variable $t = -x$ and $dt = -dx$, in the third line we use that f is even to get $f(-t) = f(t)$, and in the last step we flip sign by swapping the extrema of integration.

6. Calculate the following formal integrals.

   (a) $\int \sin(x)^2 dx$

   (b) $\int \arcsin(x)\ dx$

   (c) $\int \frac{\sinh(x)}{e^x+1}\ dx$

   (d) $\int e^{ax} \cos(bx)\, dx$ $\qquad (a \neq 0)$, (*Hint: apply integration by parts multiple times until you see a pattern.*)

Solution:

(a) We apply integration by parts with $f = \sin(x)$, $g = -\cos(x)$ and $f' = \cos(x)$, $g' = \sin(x)$.

$$\int \sin(x)^2 dx = -\sin(x)\cos(x) + \int \cos(x)^2 dx$$
$$= -\sin(x)\cos(x) + \int 1 - \sin(x)^2 dx = -\sin(x)\cos(x) + x - \int \sin(x)^2 dx.$$

From the equation we just obtained we get $\int \sin(x)^2 dx = \frac{1}{2}(x - \sin(x)\cos(x)) + C$.

(b) We apply integration by parts. Take $f'(x) = 1$ so $f(x) = x$ and $g(x) = \arcsin x$ so $g'(x) = \frac{1}{\sqrt{1-x^2}}$. We get

$$\int \arcsin x\ dx = \int 1 \cdot \arcsin x\ dx = x \cdot \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}\ dx}.$$

To figure out the last integral we notice that if we define $u(x) = (1 - x^2)$ then $u'(x) = -2x$. So we can write

$$\int x \cdot \frac{1}{\sqrt{1-x^2}}\ dx = -\frac{1}{2}\int (-2x)(1-x^2)^{-1/2}\ dx = -\frac{1}{2}\int u'(x)u(x)^{-1/2} dx$$
$$= -u(x)^{1/2} + C = -(1-x^2)^{1/2} + C = -\sqrt{1-x^2} + C.$$

Now combine everything together and write:

$$\int \arcsin x\ dx = x \cdot \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}\ dx} = x \cdot \arcsin x + \sqrt{1-x^2} + C.$$

(c) We use the definition of sinh:

$$\int \frac{\sinh(x)}{e^x+1}\, dx = \frac{1}{2}\int \frac{e^x - e^{-x}}{e^x+1}\, dx = \frac{1}{2}\int \frac{1-(e^{-x})^2}{1+e^{-x}}\, dx$$
$$= \frac{1}{2}\int (1 - e^{-x})\, dx = \frac{1}{2}\left(x + e^{-x}\right) + C\,.$$

(d) We apply integration by parts twice. Let $I_{a,b} = \displaystyle\int e^{ax}\cos(bx)\,dx$. We take $f'(x) = e^{ax}$ [$\Rightarrow f(x) = \frac{1}{a}\,e^{ax}$] and also $g(x) = \cos(bx)$ [$\Rightarrow g'(x) = -b\sin(bx)$] we get

$$I_{a,b} = \frac{1}{a}\,e^{ax}\cos(bx) + \frac{b}{a}\int e^{ax}\sin(bx)\,dx.$$

We apply integration by parts one more time on the last integral. Take $f'(x) = e^{ax}$ and $g(x) = \sin(bx)$ [$\Rightarrow g'(x) = b\cos(bx)$]

$$\int e^{ax}\sin(bx)\,dx = \frac{1}{a}\,e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos(bx)\,dx.$$

Note that we recovered $I_{a,b}$ again in last integral. So we can combine the two equations and compute $I_{a,b}$ as follows:

$$\begin{aligned} I_{a,b} &= \frac{1}{a}\,e^{ax}\cos(bx) + \frac{b}{a}\left(\frac{1}{a}\,e^{ax}\sin(bx) - \frac{b}{a}\,I_{a,b}\right) \\ \Leftrightarrow \quad \left(1 + \frac{b^2}{a^2}\right) I_{a,b} &= \frac{e^{ax}}{a}\left(\cos(bx) + \frac{b}{a}\sin(bx)\right) \\ \Leftrightarrow I_{a,b} &= \frac{e^{ax}}{a^2+b^2}\Big(a\cos(bx) + b\sin(bx)\Big) + C. \end{aligned}$$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with a period $T > 0$. Let F be defined by

$$F(x) = \int_0^x f(t)\,dt.$$

Show that F is periodic with period T if and only if

$$\int_0^T f(t)\,dt = 0.$$

Solution: We first assume F is periodic with period T, and then show this implies $\int_0^T f(t)\,dt = 0$. This is easy, we just insert the definition of F

$$0 = F(T) - F(0) = \int_0^T f(t)\,dt - \int_0^0 f(t)\,dt = \int_0^T f(t)\,dt.$$

Now we must show that $\int_0^T f(t)\,dt = 0$ implies that F is periodic i.e. $F(x+T) - F(x) = 0$ for all $x \in \mathbb{R}$. Fix one $x \in \mathbb{R}$. We can use the definition of F to write

$$F(x+T) - F(x) = \int_0^{x+T} f(t)\ dt - \int_0^x f(t)\ dt = \int_x^{x+T} f(t)\ dt = \int_x^{x+T} f(t)\ dt.$$

In the last equality, we used the fact that the integral of a periodic function with period T is always the same on any interval of length T; we will prove this claim below. But by the assumption $\int_0^T f(t)\,dt = 0$. So $F(x+T) - F(x) = 0$.

So, we are left to show that the integral of a periodic function with period T is always the same on any interval of length T. Said otherwise, we need to show that

$$\int_0^T f(s)ds = \int_x^{x+T} f(t)dt$$

for every $x \in \mathbb{R}$. First, we consider

$$k = \sup\{m \in \mathbb{Z} | mT < x + T\},$$

which exists, as the set is non-empty (e.g., take m very negative so that $mT < x$) and bounded above (e.g., bounded above by $|x|/T + 1$). Then, we use the change of variable $s = t - kT$, and we get

$$\int_x^{x+T} f(t)dt = \int_{x-kT}^{x+T-kT} f(s+kT)ds = \int_{x-kT}^{x+T-kT} f(s)ds,$$

where in the last equality we used that $f(s) = f(s + kT)$, by the T-periodicity. Now, by the definition of k, we have $x + T - kT > 0$ and $x - kT = x + T - (k + 1)T \leq 0$. So, if we define $y = x - kT$, we have $y \leq 0$ and $0 < y + T \leq T$. So, we have

$$\int_{x-kT}^{x+T-kT} f(s)ds = \int_y^{y+T} f(s)ds = \int_y^0 f(s)ds + \int_0^{y+T} f(s)ds. \tag{1}$$

Now, on the first summand, we make the change of variable $u = s + T$. So, we have

$$\int_y^0 f(s)ds = \int_{y+T}^T f(u-T)du = \int_{y+T}^T f(u)du,$$

where in the second equality we used that $f(u) = f(u - T)$. If we substitute it back in (1), we get

$$\int_{x-kT}^{x+T-kT} f(s)ds = \int_y^0 f(s)ds + \int_0^{y+T} f(s)ds = \int_{y+T}^T f(s)ds + \int_0^{y+T} f(s)ds = \int_0^T f(s)ds,$$

where we used that the variable of integration is just a dummy variable, so we can change the letter we used.

8. Calculate the following integrals.

   (a) $\int_{\pi^2/16}^{\pi^2/9} \cos(\sqrt{x})\, dx$

   (b) $\int_0^{\pi^{1/2017}} \sin(\sin(x^{2017}))\, \cos(x^{2017})\, x^{2016}\, dx$

Solution:

(a) We change the variable using $u = \sqrt{x}$ which gives $x = u^2$ and $dx = 2u\, du$. Note that $u(\pi^2/16) = \pi/4$ and $u(\pi^2/9) = \pi/3$. We have

$$\int_{\pi^2/16}^{\pi^2/9} \cos(\sqrt{x})\, dx = 2\int_{\pi/4}^{\pi/3} u\cos(u)\, du \overset{(*)}{=} 2\Big[u\sin(u)\Big]_{\pi/4}^{\pi/3} - 2\int_{\pi/4}^{\pi/3} \sin(u)\, du$$

$$= 2\Big[u\sin(u) + \cos(u)\Big]_{\pi/4}^{\pi/3} = 1 - \sqrt{2} - \frac{\pi\sqrt{2}}{4} + \frac{\pi\sqrt{3}}{3}\ .$$

For $(*)$ we used integration by parts by taking $f'(u) = \cos(u)\,,\ g(u) = u$.

(b) We change the variable with $u = x^{2017}$ which gives $du = 2017x^{2016}\ dx$. Note that $u(0) = 0$ and $u(\pi^{1/2017}) = \pi$. We get

$$\begin{aligned}\int_0^{\pi^{1/2017}} \sin(\sin(x^{2017}))\,\cos(x^{2017})\,x^{2016}\,dx &= \frac{1}{2017}\int_0^{\pi} \sin(\sin(u))\cos(u)\,du \\ &= \frac{1}{2017}\Big[-\cos(\sin(u))\Big]_0^{\pi} \qquad \text{since } \big(\sin(u)\big)' = \cos(u) \\ &= \frac{1}{2017}\Big(-\cos(\sin(\pi)) + \cos(\sin(0))\Big) \\ &= \frac{1}{2017}(-\cos(0) + \cos(0)) = 0\ .\end{aligned}$$

9. **True/False:** Let $I \subset \mathbb{R}$ be an open non-empty and bounded interval and let $f\colon I \to \mathbb{R}$ be a continuous function. Let $[a,b] \subseteq I$. If the statement is true you should prove it. If the statement is false you should give a counter example.

(a) If $\displaystyle\int_a^b f(x)\,dx = 0$, then f has a zero $[a,b]$.

(b) If $\displaystyle\int_a^b f(x)\,dx \geq 0$, then $f(x) \geq 0$ for all $x \in [a,b]$.

(c) If $f(x) < 0$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx < 0$.

Solution:

(a) True. By the mean value theorem for integrals, there exists $u \in]a,b[$ such that $0 = \int_a^b f(x)\,dx = f(u)(b-a)$. Since $b > a$, we must have $f(u) = 0$.

(b) False. Take for example $f(x) = x$ on the interval $[-1,2]$. Then $\int_{-1}^{2} f(x)\,dx = \left(\frac{x^2}{2}\right)\Big|_{-1}^{2} = \frac{3}{2} \geq 0$ but $f(-1) = -1 < 0$.

(c) True. By the mean value theorem for integrals, there exists $u \in]a,b[$ such that $\int_a^b f(x)\,dx = f(u)(b-a)$. Since we have $f(u) < 0$ and $b > a$, the result holds.

10. (a) Show that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

(b) Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ \frac{1}{2}, & x \notin \mathbb{Q}. \end{cases}$$

Compute the upper and lower Darboux sums for the regular partitions σ_{2n}. Is f integrable?

Solution:

(a) We prove it by induction. If $n = 0$ or 1 the formula holds. Assume that the formula holds for $n-1$, then

$$\sum_{i=1}^{n} i = n + \sum_{i=1}^{n-1} i = n + \frac{n(n-1)}{2} = \frac{2n + n^2 - n}{2} = \frac{n(n+1)}{2}.$$

(b) The regular partition σ_{2n} of the interval $[0,1]$ is $0, \frac{1}{2n}, \frac{2}{2n}, \dots, \frac{2n-1}{2n}, \frac{2n}{2n} = 1$. So

$$\underline{S}_{\sigma_{2n}} = \sum_{i=1}^{2n} \left(\inf_{x \in [\frac{i-1}{2n}, \frac{i}{2n}]} f(x) \right) \left(\frac{i}{2n} - \frac{i-1}{2n} \right), \quad \overline{S}_{\sigma_{2n}} = \sum_{i=1}^{2n} \left(\sup_{x \in [\frac{i-1}{2n}, \frac{i}{2n}]} f(x) \right) \left(\frac{i}{2n} - \frac{i-1}{2n} \right).$$

We observe that $\frac{i}{2n} - \frac{i-1}{2n} = \frac{1}{2n}$ for all i. Also,

$$\inf_{x \in [\frac{i-1}{2n}, \frac{i}{2n}]} f(x) = \begin{cases} \frac{i-1}{2n} & \text{if } i \le n \\ \frac{1}{2} & \text{if } i > n \end{cases}, \qquad \sup_{x \in [\frac{i-1}{2n}, \frac{i}{2n}]} f(x) = \begin{cases} \frac{1}{2} & \text{if } i \le n \\ \frac{i}{2n} & \text{if } i > n \end{cases}$$

So we compute

$$\begin{aligned}
\underline{S}_{\sigma_{2n}} &= \left(\sum_{i=1}^{n} \frac{i-1}{2n} \cdot \frac{1}{2n} \right) + \left(\sum_{i=n+1}^{2n} \frac{1}{2} \cdot \frac{1}{2n} \right) = \frac{1}{4n^2} \left(\left(\sum_{i=1}^{n} i \right) - \sum_{i=1}^{n} 1 \right) + \frac{1}{4n} \sum_{i=n+1}^{2n} 1 \\
&= \frac{1}{4n^2} \left(\frac{n(n+1)}{2} - n \right) + \frac{1}{4} = \frac{3}{8} - \frac{1}{8n} \\
\overline{S}_{\sigma_{2n}} &= \left(\sum_{i=1}^{n} \frac{1}{2} \cdot \frac{1}{2n} \right) + \left(\sum_{i=n+1}^{2n} \frac{i}{2n} \cdot \frac{1}{2n} \right) = \frac{1}{4} + \frac{1}{4n^2} \left(\left(\sum_{i=1}^{2n} i \right) - \left(\sum_{i=1}^{n} i \right) \right) \\
&= \frac{1}{4} + \frac{1}{4n^2} \left(\frac{2n(2n+1)}{2} - \frac{n(n+1)}{2} \right) = \frac{5}{8} + \frac{1}{8n}.
\end{aligned}$$

We observe that the sequences $(\underline{S}_{\sigma_{2n}})$ and $(\overline{S}_{\sigma_{2n}})$ converge to different limits, so we expect that the function is not integrable. But finding two converging sequences of Darboux sums that do not have the same limit is not enough to prove that $\underline{S} \neq \overline{S}$, because the sup and inf are taken over all the partition and partitions like $0 = x_0 < \cdots < x_i = \frac{1}{\sqrt{2}} < \dots x_m = 1$ cannot be refined by a regular partition because $x_i \notin \mathbb{Q}$.

First method. One way to proceed is by looking at the graph of the function f, which looks like the union of the graphs of the two functions

$$\underline{f}(x) = \begin{cases} x, & x \in [0, \frac{1}{2}] \\ \frac{1}{2}, & x \in [\frac{1}{2}, 1] \end{cases} \qquad \overline{f}(x) = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}] \\ x, & x \in [\frac{1}{2}, 1] \end{cases}$$

and show that for every partition σ we have that $\underline{S}_\sigma$ is smaller that the area under the graph of $\underline{f}$, and that $\overline{S}_\sigma$ is bigger that the area under the graph of $\overline{f}$, since these two areas are different $\underline{S}$ and $\overline{S}$ cannot be equal.

Second method. Another way to prove that f is not integrable is by comparing arbitrary partitions with the regular partitions that we computed above and compute the Darboux integrals $\underline{S}$ and $\overline{S}$. Recall that if σ, τ are two partitions and σ is a refinement of τ, then $\underline{S}_\sigma \geq \underline{S}_\tau$ and $\overline{S}_\sigma \leq \overline{S}_\tau$. So given an arbitrary partition τ it is enough to make the computations for its refinements, in particular, we can choose a partition σ that refined both τ and σ_{2n} for some n.

Given a partition σ that refines σ_{2n}, we can write σ as a collection of partitions

$$\begin{aligned} 0 = x_{1,0}, \dots, x_{1,m_1} &= \frac{1}{2n} \quad \text{ of } \quad [0, \frac{1}{2n}] \\ &\vdots \\ \frac{i-1}{2n} = x_{i,0}, \dots, x_{i,m_i} &= 1 \qquad \text{of } [\frac{i-1}{2n}, \frac{1}{2n}] \\ &\vdots \\ \frac{2n-1}{2n} = x_{2n,0}, \dots, x_{2n,m_{2n}} &= 1 \qquad \text{of } [\frac{2n-1}{2n}, 1] \end{aligned}$$

then

$$\begin{aligned} \underline{S}_\sigma &= \left(\sum_{i=1}^{n} \sum_{j=1}^{m_i} x_{i,j-1}(x_{i,j} - x_{i,j-1}) \right) + \left(\sum_{i=n+1}^{2n} \sum_{j=1}^{m_i} \frac{1}{2}(x_{i,j} - x_{i,j-1}) \right) \\ &\leq \left(\sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{i}{2n}(x_{i,j} - x_{i,j-1}) \right) + \left(\sum_{i=n+1}^{2n} \sum_{j=1}^{m_i} \frac{1}{2}(x_{i,j} - x_{i,j-1}) \right) \\ &= \left(\sum_{i=1}^{n} \frac{i}{2n} \cdot \frac{1}{2n} \right) + \left(\sum_{i=n+1}^{2n} \frac{1}{2} \cdot \frac{1}{2n} \right) = \underline{S}_{\sigma_{2n}} + \sum_{i=1}^{n} \frac{1}{2n} \cdot \frac{1}{2n} = \underline{S}_{\sigma_{2n}} + \frac{1}{4n} \end{aligned}$$

So

$$\underline{S} = \sup_\sigma \underline{S}_\sigma \leq \sup_{n \geq 2} \left(\underline{S}_{\sigma_{2n}} + \frac{1}{4n} \right) = \sup_{n \geq 2} \left(\frac{3}{8} + \frac{1}{8n} \right) = \frac{3}{8} + \frac{1}{16} = \frac{7}{16}$$

Similarly,

$$\begin{aligned} \overline{S}_\sigma &= \left(\sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{1}{2}(x_{i,j} - x_{i,j-1}) \right) + \left(\sum_{i=n+1}^{2n} \sum_{j=1}^{m_i} x_{i,j}(x_{i,j} - x_{i,j-1}) \right) \\ &\geq \left(\sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{1}{2}(x_{i,j} - x_{i,j-1}) \right) + \left(\sum_{i=n+1}^{2n} \sum_{j=1}^{m_i} \frac{i-1}{2n}(x_{i,j} - x_{i,j-1}) \right) \\ &= \left(\sum_{i=1}^{n} \frac{1}{2} \cdot \frac{1}{2n} \right) + \left(\sum_{i=n+1}^{2n} \frac{i-1}{2n} \cdot \frac{1}{2n} \right) = \overline{S}_{\sigma_{2n}} - \sum_{i=1}^{n} \frac{1}{2n} \cdot \frac{1}{2n} = \overline{S}_{\sigma_{2n}} - \frac{1}{4n} \end{aligned}$$

and

$$\overline{S} = \inf_\sigma \overline{S}_\sigma \geq \inf_{n \geq 2} \left(\overline{S}_{\sigma_{2n}} - \frac{1}{4n} \right) = \inf_{n \geq 2} \left(\frac{5}{8} - \frac{1}{8n} \right) = \frac{5}{8} - \frac{1}{16} = \frac{9}{16} > \underline{S}$$

Since $\underline{S} \neq \overline{S}$ we conclude that f is not integrable.

11. (a) Show that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.
    (b) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 2x^2 + 3x - 1$ Compute the upper and lower Darboux sums for the regular partitions σ_n. Is f integrable?

Solution:

(a) We prove it by induction. If $n = 1$ the formula holds by direct inspection. Assume that the formula holds for $n-1 \geq 1$, then

$$\sum_{i=1}^{n} i^2 = n^2 + \sum_{i=1}^{n-1} i^2 = n^2 + \frac{n(n-1)(2n-1)}{6} = \frac{6n^2 + n(2n^2 - 3n + 1)}{6} = \frac{n(2n^2 + 3n + 1)}{6}.$$

(b) The regular partition σ_n of the interval $[0, 1]$ is $0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} = 1$. We observe that the function f is increasing because $f'(x) = 4x + 3$ is ≥ 0 on $[0, 1]$. Then we have

$$\begin{aligned}
\underline{S}_{\sigma_n} &= \sum_{i=1}^{n} \left(\inf_{x \in [\frac{i-1}{n}, \frac{i}{n}]} f(x) \right) \left(\frac{i}{n} - \frac{i-1}{n} \right) = \sum_{i=1}^{n} \left(2\left(\frac{i-1}{n}\right)^2 + 3\frac{i-1}{n} - 1 \right) \frac{1}{n} \\
&= \frac{2}{n^3} \sum_{i=1}^{n} (i-1)^2 + \frac{3}{n^2} \sum_{i=1}^{n} (i-1) - \frac{1}{n} \sum_{i=1}^{n} 1 = \left(\frac{2}{n^3} \sum_{j=1}^{n-1} j^2 \right) + \left(\frac{3}{n^2} \sum_{j=1}^{n-1} j \right) - 1 \\
&= \frac{2}{n^3} \frac{n(n-1)(2n-1)}{6} + \frac{3}{n^2} \frac{n(n-1)}{2} - 1 = \frac{7}{6} - \frac{15}{6n} + \frac{3}{2n^2} \\
\overline{S}_{\sigma_n} &= \sum_{i=1}^{n} \left(\sup_{x \in [\frac{i-1}{n}, \frac{i}{n}]} f(x) \right) \left(\frac{i}{n} - \frac{i-1}{n} \right) = \sum_{i=1}^{n} \left(2\left(\frac{i}{n}\right)^2 + 3\frac{i}{n} - 1 \right) \frac{1}{n} \\
&= \frac{2}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n^2} \sum_{i=1}^{n} i - \frac{1}{n} \sum_{i=1}^{n} 1 = \frac{2}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n^2} \frac{n(n+1)}{2} - 1 \\
&= \frac{7}{6} + \frac{15}{6n} + \frac{3}{2n^2}
\end{aligned}$$

We observe that $\lim_{n \to +\infty} \underline{S}_{\sigma_n} = \lim_{n \to +\infty} \overline{S}_{\sigma_n} = \frac{7}{6}$. So the function f is integrable.

12. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) If $f(x) = x + e^x$, then $(f^{-1})'(1) = 1 + \frac{1}{e}$.

(b) If f is differentiable on the interval $I \subset \mathbb{R}$, then f' is continuous on I.

Solution:

(a) False.
The formula for the derivative of the inverse function is $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.
Here we have $f'(x) = 1 + e^x$ and $f^{-1}(1) = 0$ since $f(0) = 1$. So

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{2}.$$

(b) False.
Take for example the function f, $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$. This function is

differentiable at all $x \neq 0$, as it is the composition of differentiable functions. To check the differentiability at $x = 0$, we claim that $f'(0) = 0$:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \cos\left(\frac{1}{x}\right) - 0}{x - 0} = \lim_{x\to 0} \frac{x^2 \cos\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} x \cos\left(\frac{1}{x}\right) = 0$$

So we can write the derivative of f as

$$f(x) = \begin{cases} 2x\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We see that f is differentiable on $]-1, 1[$ (in fact on $\mathbb{R}$) but the derivative is not continuous at $x = 0$.