Analysis 1 - Exercise Set 13

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. State on which closed intervals the following functions are integrable and compute the antiderivatives.
   (a) $f(x) = e^x$;
   (b) $f(x) = \sinh(x)$;
   (c) $f(x) = (ax + b)^s$ with $s \in \mathbb{Z}$ and $a, b \in \mathbb{R} \setminus \{0\}$;
   (d) $f(x) = \cos(x)^3$;
   (e) $f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0; \end{cases}$
   (f) $f(x) = \cot(x)$, $\cot(x) := \frac{\cos(x)}{\sin(x)}$.
   (g) $f(x) = |x|^s$, $s > 0$.

2. Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 + 3x + 2$ on the interval $[1, 4]$.

3. Let
$$f(x) = \begin{cases} \sin(x) & 0 \leq x \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \leq x \leq 3 \end{cases}$$
   Compute $\int_0^3 f(x)dx$.

4. **True/False:** If the statement is true you should prove it. If it is false you should give a counterexample. Let F be an anti-derivative of f on $[a, b]$.
   (a) If $f(x) \leq 0$ for all $x \in [a, b]$, then $F(x) \leq 0$ for all $x \in [a, b]$.
   (b) For all $x \in [a, b]$, we have $F(x) = \displaystyle\int_a^x f(t)\, dt$.

5. Show that:
   (a) if $f : [-a, a] \to \mathbb{R}$ is an integrable odd function then $\int_{-a}^a f(x)dx = 0$;
   (b) if $f : [-a, a] \to \mathbb{R}$ is an integrable even function then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.

6. Calculate the following formal integrals.
   (a) $\int \sin(x)^2 dx$
   (b) $\int \arcsin(x)\, dx$
   (c) $\int \frac{\sinh(x)}{e^x+1}\, dx$
   (d) $\int e^{ax} \cos(bx)\, dx$ $\quad (a \neq 0)$, (*Hint: apply integration by parts multiple times until you see a pattern.*)

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with a period $T > 0$. Let F be defined by

$$F(x) = \int_0^x f(t)\, dt.$$

Show that F is periodic with period T if and only if

$$\int_0^T f(t)\, dt = 0.$$

8. Calculate the following integrals.

   (a) $\int_{\pi^2/16}^{\pi^2/9} \cos(\sqrt{x})\, dx$

   (b) $\int_0^{\pi^{1/2017}} \sin(\sin(x^{2017}))\cos(x^{2017})\, x^{2016}\, dx$

9. **True/False:** Let $I \subset \mathbb{R}$ be an open non-empty and bounded interval and let $f\colon I \to \mathbb{R}$ be a continuous function. Let $[a, b] \subseteq I$. If the statement is true you should prove it. If the statement is false you should give a counter example.

   (a) If $\displaystyle\int_a^b f(x)\, dx = 0$, then f has a zero $[a, b]$.

   (b) If $\displaystyle\int_a^b f(x)\, dx \geq 0$, then $f(x) \geq 0$ for all $x \in [a, b]$.

   (c) If $f(x) < 0$ for all $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\, dx < 0$.

10. (a) Show that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

    (b) Let $f : [0, 1] \to \mathbb{R}$ be defined by

    $$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ \frac{1}{2}, & x \notin \mathbb{Q}. \end{cases}$$

    Compute the upper and lower Darboux sums for the regular partitions σ_{2n}. Is f integrable?

11. (a) Show that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

    (b) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 2x^2 + 3x - 1$ Compute the upper and lower Darboux sums for the regular partitions σ_n. Is f integrable?

12. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

    (a) If $f(x) = x + e^x$, then $(f^{-1})'(1) = 1 + \frac{1}{e}$.

    (b) If f is differentiable on the interval $I \subset \mathbb{R}$, then f' is continuous on I.