Analysis 1 - Exercise Set 12

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Calculate the following limits:

   (a) $\lim_{x\to 0} \big(1+\sin(x)\big)^{1/x}$

   (Hint: Write $\big(1+\sin(x)\big)^{1/x} = e^{\left(\frac{1}{x}\log\left(1+\sin(x)\right)\right)}$ *and first calculate the limit of the exponent.)*

   (b) $\lim_{x\to\sqrt{3}} \frac{x^x-\sqrt{3}^{\sqrt{3}}}{x-\sqrt{3}}$. Remember that $x^x = e^{x\log(x)}$.

2. Find the Taylor expansion of order 5 at $x=0$ of the following functions.

   (a) $f(x) = \sin(x)$
   (b) $f(x) = \log(1+x)$
   (c) $f(x) = \tan(x)$
   (d) $f(x) = \arccos(x)$
   (e) $f(x) = \sinh(x)$
   (f) $f(x) = \log(\cos(x))$

3. Use Taylor expansion to find the following limits.

   (a) $\lim_{x\to 0} \dfrac{x-\frac{x^3}{6}-\sin(x)}{x^5}$

   (b) $\lim_{x\to 0} \dfrac{e^x+\sin(x)-\cos(x)-2x}{x-\log(1+x)}$

   (c) $\lim_{x\to 0} \dfrac{x\sin(\sin(x))-\sin(x)^2}{x^6}$

   (d) $\lim_{x\to 0} \frac{\sqrt[3]{1-x}-1}{\sqrt[4]{1-x}-1}$

4. Calculate the following limits:

   (a) $\lim_{x\to+\infty} x\left(\tanh(x)-1\right)$

   (b) $\lim_{x\to+\infty} \frac{e^x}{x^n}$ where $n\in\mathbb{N}$. First find the limit using $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ and then using l'Hopital's rule.

   (c) $\lim_{x\to 0^+} x^n\log(x)$

   (d) $\lim_{x\to\infty} \frac{x^n}{\log(x)}$

5. Calculate the derivative f' of the function $f(x) = \log_3\big(\cosh(x)\big)$ and give the domain of f and f'.

6. Calculate the following limit $\lim_{x\to 2} \dfrac{\log(x-1)}{x-2}$.

7. Let the function $f :]-\pi/2, \pi/2[\to \mathbb{R}$ be defined by $f(x) = \log(1+\sin(x))$. What is the Taylor expansion of order 3 of f at $x = 0$.

   (a) $x - \frac{x^3}{6} + \cdots$
   (b) $x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$
   (c) $x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots$
   (d) $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

8. Find the Taylor expansion of order n at $x = 0$ of the following functions.

   (a) $f(x) = e^{\sin(x)}, \quad n = 4$
   (b) $f(x) = \sqrt{1+\sin(x)}, \quad n = 3$

9. For each one of the following functions, determine whether the funciton is differentiable at $x = 0$. If yes, also compute the derivative at $x = 0$:

   (a) $f(x) = \begin{cases} x+1, & x \geq 0 \\ x, & x < 0 \end{cases};$

   (b) $f(x) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases};$

   (c) $f(x) = \begin{cases} \frac{\sin(x)-x}{x}, & x > 0 \\ 0, & x = 0 \\ \frac{\cos(x)-\frac{x^2}{2}}{x^4}, & x < 0 \end{cases}.$

10. State if the following statements are true or false. Let $f, g : I \to \mathbb{R}$ be two convex functions, where $I \subset \mathbb{R}$ is some interval. If it is true, prove it. If not, give a counter example.

    (a) The function $f + g$ is convex.
    (b) The function $h = f \cdot g$ is convex.
    (c) If g is increasing then the function $h = g \circ f$ is convex.

11. Consider $f\colon]a, b[\mapsto \mathbb{R}$. Let $g\colon]c, d[\mapsto \mathbb{R}$ be the restriction to f to the interval $]c, d[\subset]a, b[$, i.e., $f(x) = g(x) \quad \forall x \in]c, d[$. Show that

    (a) If $f \in C^n(]a, b[, \mathbb{R})$ then $g \in C^n(]c, d[, \mathbb{R})$.
    (b) If f is Lipschitz continuous, then g is Lipschitz continuous.

12. Find the local extrema and the absolute maximum and minimum of $f(x) = x^2 - \left|x + \frac{1}{4}\right| + 1$ in $[-1, 1]$.

13. Let $a, b \in \overline{\mathbb{R}}$, $a < b$. Let $f :]a, b[\to \mathbb{R}$ be a differentiable function. State if the following statements are true or false. If it is true, prove it. If not, give a counter example.

    (a) If f' is bounded, then f is Lipschitz continuous with Lipschitz contstant $k = \sup_{x\in]a,b[} |f'(x)|$.
    (b) If f is Lipschitz continuous, then it is uniformly continuous.
    (c) If f' is bounded then f is uniformly continuous.

14. Study the function $f(x) = \dfrac{x}{x^2-1}$ and sketch its graph (domain, range, symmetries, roots, continuity, differentiability, stationary points, extrema, convexity, inflection points, asymptotes).

15. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = e^x$. Compute the upper and lower Darboux sums for the regular partitions σ_n. Is f integrable?

16. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function on $[a,b] \subset D(f)$, $a < b$, and differentiable on $]a,b[$.

    (a) If $f'(x) \geq 0$ for all $x \in]a,b[$, then f is increasing on $[a,b]$.

    (b) If f is increasing on $[a,b]$, then $f'(x) \geq 0$ for all $x \in]a,b[$.

    (c) If f is strictly increasing on $[a,b]$, then $f'(x) > 0$ for all $x \in]a,b[$.

    (d) If $f'(x) > 0$ for all $x \in]a,b[$, then f is strictly increasing on $[a,b]$.

    (e) If $\lim\limits_{x\to a^+} f'(x) = \ell$ exists, then f is differentiable from right at a and the right derivative is $f'_d(a) = \ell$.

17. Using the definition of convex functions, show that the function $f(x) = x^2$ is convex.

18. Find the local extrema and the absolute maximum and minimum of $f(x) = (x-1)^2 - 2\,|2-x|$ in $]2,3[$

19. Study the function $f(x) = \dfrac{3x^2 - x}{2x - 1}$ and sketch its graph (domain, range, symmetries, roots, continuity, differentiability, stationary points, extrema, convexity, inflection points, asymptotes).

20. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathbb{R}$ with $g'(x) \neq 0$ for all $x \in \mathbb{R}$.

    (a) If $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$, then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)}$.

    (b) If $\lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist, then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ does not exist.

21. Using the definition of convex functions, show that the function $f(x) = \frac{1}{x}$, $\quad x \in]0, +\infty[$ is convex.