Analysis 1 - Exercise Set 12

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Calculate the following limits:

   (a) $\lim_{x\to 0}\big(1+\sin(x)\big)^{1/x}$

   (*Hint: Write* $\big(1+\sin(x)\big)^{1/x}=e^{\left(\frac{1}{x}\log\left(1+\sin(x)\right)\right)}$ *and first calculate the limit of the exponent.*)

   (b) $\lim_{x\to\sqrt{3}}\frac{x^x-\sqrt{3}^{\sqrt{3}}}{x-\sqrt{3}}$. Remember that $x^x=e^{x\log(x)}$.

Solution:

For calculating these limits we apply the L'Hôpital rule, but we must check the hypothesis in each case:

(a) We have $\big(1+\sin(x)\big)^{1/x}=e^{\frac{1}{x}\log\left(1+\sin(x)\right)}$. We are first going to calculate the limit of the exponent. Take $f(x)=\log\big(1+\sin(x)\big)$ and $g(x)=x$. Then $\lim_{x\to 0}f(x)=\lim_{x\to 0}g(x)=0$ and $g'(x)=1\neq 0$. So

$$\lim_{x\to 0}\frac{\log(1+\sin(x))}{x}\overset{\text{L'Hôpital}}{=}\lim_{x\to 0}\frac{\frac{\cos(x)}{1+\sin(x)}}{1}=1\,,$$

Finally, by the continuity of the exponential function we have

$$\lim_{x\to 0}(1+\sin(x))^{1/x}=e^1=e\,.$$

We may apply L'Hôpital's rule since $(x)'\neq 0\quad\forall$ intervals I containing 0.

(b) Take $f(x)=x^x-\sqrt{3}^{\sqrt{3}}$ and $g(x)=x-\sqrt{3}$. Note that we can rewrite $f(x)=e^{x\log x}-e^{\sqrt{3}\log\sqrt{3}}$. We have that $\lim_{x\to\sqrt{3}}f=\lim_{x\to\sqrt{3}}g=0$ and that $g'(x)=1\neq 0$. So the L'Hôpital rule applies. We have

$$\begin{aligned}\lim_{x\to\sqrt{3}}\frac{x^x-\sqrt{3}^{\sqrt{3}}}{x-\sqrt{3}}&=\lim_{x\to\sqrt{3}}\frac{e^{x\log x}-e^{\sqrt{3}\log\sqrt{3}}}{x-\sqrt{3}}\\&\overset{\text{L'Hôpital}}{=}\lim_{x\to\sqrt{3}}\frac{(1+\log x)e^{x\log x}}{1}\\&=(1+\frac{1}{2}\log 3)\sqrt{3}^{\sqrt{3}}\end{aligned}$$

We may apply L'Hôpital's rule since $(x-\sqrt{3})'\neq 0\quad\forall$ intervals I containing 0.

2. Find the Taylor expansion of order 5 at $x = 0$ of the following functions.

   (a) $f(x) = \sin(x)$
   (b) $f(x) = \log(1 + x)$
   (c) $f(x) = \tan(x)$
   (d) $f(x) = \arccos(x)$
   (e) $f(x) = \sinh(x)$
   (f) $f(x) = \log(\cos(x))$

Solution:

The Taylor's expansion of order 5 around the point a of a function f is given by the

$$\begin{aligned} f(x) = f(a) &+ \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &+ \frac{f^{(4)}(a)}{4!}(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5 + \varepsilon_5(x) \end{aligned}$$

where $\varepsilon_5(x) = \dfrac{f^{(6)}(u)}{6!}(x-a)^6$ for a certain u between a and x, we can say that $u \in]a, x[$ if $x > a$ and that $u \in]x, a[$ if $x < a$.

(a) For $\sin x$ we notice that all the even derivatives are zero at $x = 0$ and odd derivatives are $\sin^{(4k+1)}(x) = \cos(x)$ and that $\cos(0) = 1$ and $\sin^{(4k++3)}(x) = -\cos(x)$ and that $-\cos(0) = -1$. If we plug in all the derivatives in the Taylor formula we get

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \varepsilon_5(x)$$

(b) For $\log(1 + x)$ we see that the n-th derivative is given by

$$\log^{(n)}(1+x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

So we can now use the Taylor expansion to write

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + x^5\varepsilon(x)$$

(c) By differentiating $f(x) = \tan(x)$ 5 times we see that

$$\begin{aligned} &f(0) = 0 \\ &f'(0) = 1 \\ &f''(0) = 0 \\ &f'''(0) = 2 \\ &f^{(4)} = 0 \\ &f^{(5)} = 16 \end{aligned}$$

which yields $f(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

(d) By differentiating $f(x) = \arccos(x)$ 5 times we see that

$$\begin{aligned} f(0) &= \frac{\pi}{2} \\ f'(0) &= -1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f^{(4)} &= 0 \\ f^{(5)} &= -9 \end{aligned}$$

which yields $f(x) = \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 \cdots$

(e) By differentiating $f(x) = \sinh(x)$ (note that $f^{(2k)} = \sinh(x)$ and $f^{(2k+1)}(x) = \cosh(x)$) 5 times we see that

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= 1 \\ f^{(4)} &= 0 \\ f^{(5)} &= 1 \end{aligned}$$

which yields $f(x) = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots$

(f) Note that $f'(x) = -\tan(x)$. Then we can use (c) to see that

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f'''(0) &= 0 \\ f^{(4)} &= -2 \\ f^{(5)} &= 0 \end{aligned}$$

which yields $f(x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 \cdots$

3. Use Taylor expansion to find the following limits.

(a) $\lim_{x\to 0} \dfrac{x - \frac{x^3}{6} - \sin(x)}{x^5}$

(b) $\lim_{x\to 0} \dfrac{e^x + \sin(x) - \cos(x) - 2x}{x - \log(1+x)}$

(c) $\lim_{x\to 0} \dfrac{x\sin(\sin(x)) - \sin(x)^2}{x^6}$

(d) $\lim_{x\to 0} \frac{\sqrt[3]{1-x}-1}{\sqrt[4]{1-x}-1}$

Solution: We need to choose the order of the Taylor expansion such that the indeterminacies in the denominator are eliminated. Since we are only interested in the limit, it is enough to keep higher terms as $(x-a)^n\varepsilon(x)$, where $\lim\limits_{x\to a}\varepsilon(x)=0$. In particular, if we consider a Taylor expansion of order n and have an error $\varepsilon_n(x)=\frac{f^{(n+1)}(u)}{n!}(x-a)^{n+1}$, we want to write it as $(x-a)^n\cdot\varepsilon(x)$, where we set $\varepsilon(x)=\frac{f^{(n+1)}(u)}{n!}(x-a)$. Then, as $\frac{f^{(n+1)}(u)}{n!}$ is bounded in a neighborhood of a (we are using that $f^{(n+1)}$ is continuous and u is a value, depending on x and a, that is in between a and x) and $x-a\to 0$ as $x\to a$, we get that $\varepsilon(x)\to 0$ as $x\to a$.

(a) Since

$$\sin(x)=x-\frac{x^3}{6}+\frac{x^5}{120}+x^5\varepsilon(x)\,,$$

we have

$$\lim_{x\to 0}\frac{1}{x^5}\left(x-\frac{x^3}{6}-\sin(x)\right)=\lim_{x\to 0}\left(-\frac{1}{120}+\varepsilon(x)\right)=-\frac{1}{120}\,,$$

because $\varepsilon(x)\to 0$ when $x\to 0$.

(b) We have

$$e^x+\sin(x)-\cos(x)-2x=1+x+\frac{x^2}{2}+x-1+\frac{x^2}{2}-2x+x^2\varepsilon(x)=x^2+x^2\varepsilon(x)$$

and

$$x-\log(1+x)=x-x+\frac{x^2}{2}+x^2\varepsilon(x)=\frac{x^2}{2}+x^2\tilde{\varepsilon}(x),$$

where $\varepsilon(x)$ (resp. $\tilde{\varepsilon}(x)$) goes to 0 as $x\to 0$, as it has the form $C(u(x))x^3$, where $C(x)$ is the derivative of order 3 of $e^x+\sin(x)-\cos(x)-2x$ (resp. $x-\log(1+x)$). Thus, we have

$$\lim_{x\to 0}\frac{e^x+\sin(x)-\cos(x)-2x}{x-\log(1+x)}=\lim_{x\to 0}\frac{x^2+x^2\varepsilon(x)}{\frac{x^2}{2}+x^2\varepsilon(x)}=\lim_{x\to 0}\frac{1+\varepsilon(x)}{\frac{1}{2}+\varepsilon(x)}=2.$$

(c) We need to choose the order of the Taylor expansion such that the indeterminacies in the denominator are eliminated. Since we are only interested in the limit, it is enough to keep higher terms as $(x-a)^n\varepsilon(x)$, where $\lim\limits_{x\to a}\varepsilon(x)=0$.

For the 6-th order Taylor expansion of the numerator, we need to first find the 5-th order Taylor expansion of $\sin(\sin(x))$ and then the 6-th order of $\sin(x)^2$.

Since $\sin(x)=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+x^5\varepsilon(x)$, we have

$$\sin(\sin(x))=\sin(x)-\frac{\sin(x)^3}{3!}+\frac{\sin(x)^5}{5!}+\underbrace{\sin(x)^5\varepsilon\big(\sin(x)\big)}_{=x^5\varepsilon(x)}. \tag{1}$$

In the computation, we replace $\sin(x)^5\varepsilon\big(\sin(x)\big)$ with $x^5\varepsilon(x)$. We can do this because $\sin(0)=0$ and $\lim\limits_{x\to 0}\varepsilon(x)=0$, we have $\lim\limits_{x\to 0}\varepsilon\big(\sin(x)\big)=0$ and so $\varepsilon\big(\sin(x)\big)$ behaves as $\varepsilon(x)$. Then, $\frac{\sin(x)}{x}$ is bounded in the neighborhood of 0 (as its limit as $x\to 0$ is 1) and indeed $\sin(x)^5\varepsilon(x)=\frac{\sin(x)^5}{x^5}x^5\varepsilon(x)$ behaves like $x^5\varepsilon(x)$.

For the powers of $\sin(x)$ we have

$$\sin(x)^2=\left(x-\frac{x^3}{6}+\frac{x^5}{120}+x^6\varepsilon(x)\right)^2=x^2-\frac{x^4}{3}+\frac{2x^6}{45}+x^6\varepsilon(x),$$

$$\sin(x)^3=\left(x^2-\frac{x^4}{3}+x^5\varepsilon(x)\right)\left(x-\frac{x^3}{6}+\frac{x^5}{120}+x^5\varepsilon(x)\right)=x^3-\frac{x^5}{2}+x^5\varepsilon(x),$$

$$\sin(x)^5=\left(x^2-\frac{x^4}{3}+x^5\varepsilon(x)\right)\left(x^3-\frac{x^5}{2}+x^5\varepsilon(x)\right)=x^5+x^5\varepsilon(x),$$

4. Calculate the following limits:
   (a) $\lim\limits_{x\to+\infty} x\big(\tanh(x)-1\big)$
   (b) $\lim\limits_{x\to+\infty} \frac{e^x}{x^n}$ where $n\in\mathbb{N}$. First find the limit using $e^x=\sum\limits_{k=0}^{\infty}\frac{x^k}{k!}$ and then using l'Hopital's rule.
   (c) $\lim\limits_{x\to 0^+} x^n\log(x)$
   (d) $\lim\limits_{x\to\infty} \frac{x^n}{\log(x)}$

Solution:

(a) Here we should apply the L'Hôpital rule several times. For the first time, we have $f(x)=\tanh(x)-1$ and $g(x)=\frac{1}{x}$. Since $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty} g(x)=0$ and $g'(x)=-\frac{1}{x^2}\neq 0$, the hypothesis is satisfied and we can apply the L'Hôpital rule (the hypothesis for the second and third iterations of L'Hôpital rule will be verified below) :

$$\lim_{x\to+\infty} x\big(\tanh(x)-1\big)=\lim_{x\to+\infty}\frac{\tanh(x)-1}{\frac{1}{x}}\overset{\text{L'Hôpital}}{=}\lim_{x\to+\infty}\frac{\frac{1}{\cosh(x)^2}}{-\frac{1}{x^2}}=-\lim_{x\to+\infty}\frac{x^2}{\cosh(x)^2}$$

$$\overset{\text{L'Hôpital}}{=}-\lim_{x\to+\infty}\frac{2x}{\sinh(2x)}\overset{\text{L'Hôpital}}{=}-\lim_{x\to+\infty}\frac{2}{2\cosh(2x)}=0\,.$$

To apply the L'Hôpital rule a second time we have $\tilde{f}(x)=x^2$ and $\tilde{g}(x)=\cosh(x)^2$ we have $\lim\limits_{x\to+\infty}\tilde{f}(x)=\lim\limits_{x\to+\infty}\tilde{g}(x)=+\infty$ and $g'(x)=2\sinh(x)\cosh(x)=\sinh(2x)\neq 0$ for $x\neq 0$ (which is the case when $x\to+\infty$).
Finally for the third time with $\bar{f}(x)=2x$ and $\bar{g}(x)=\sinh(2x)$ and so $\lim\limits_{x\to+\infty}\bar{f}(x)=\lim\limits_{x\to+\infty}\bar{g}(x)=+\infty$ and that $\bar{g}'(x)=2\cosh(2x)\neq 0$. And so we can apply the L'Hôpital rule for the third time.

(b) Here we need to apply the L'Hôpital rule n times. We notice that every time we apply the L'Hôpital rule the numerator remains e^x and the denominator will be of the form mx^k for some $m\in\mathbb{N}$ and $0\geq k\leq n$. So take $f(x)=e^x$ and $g(x)=mx^k$. We have that $\lim_{x\to+\infty} f=\lim_{x\to+\infty} g=+\infty$ and that $g'(x)\neq 0$. So we can apply L'Hôpital rule n times. We have

$$\lim_{x\to+\infty}\frac{e^x}{x^n}=\lim_{x\to+\infty}\frac{e^x}{nx^{n-1}}\overset{\text{L'Hôpital}}{=}\dots\overset{\text{L'Hôpital}}{=}\lim_{x\to+\infty}\frac{e^x}{n!}=+\infty$$

Now using the Taylor series expansion we see that $\frac{e^x}{x^n}=\delta(x)+\sum\limits_{k=n}^{\infty}\frac{x^{k-n}}{k!}=\delta(x)+\sum\limits_{k=0}^{\infty}\frac{x^k}{(k+n)!}\geq\delta(x)+\frac{1}{k!}+\frac{1}{(k+1)!}x\to+\infty$ when $x\to+\infty$ where $\delta(x)$ is a function s.t. $\delta(x)\to 0$ as $x\to+\infty$.

(c)

$$\lim_{x\to 0^+} x^n\log(x)=\lim_{x\to 0^+}\frac{\log(x)}{\frac{1}{x^n}}\overset{\text{L'Hôpital}}{=}\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{n}{x^{n+1}}}=0$$

(d) $\lim\limits_{x\to+\infty}\frac{x^n}{\log(x)}=\lim\limits_{h\to 0^+}\frac{1}{-h^n\log(h)}=+\infty$ by using (c).

5. Calculate the derivative f' of the function $f(x) = \log_3\big(\cosh(x)\big)$ and give the domain of f and f'.

Solution:
We observe that

$$f(x) = \frac{\log\big(\cosh(x)\big)}{\log(3)}\ .$$

So $\quad f'(x) = \dfrac{\sinh(x)}{\log(3)\cosh(x)} = \dfrac{\tanh(x)}{\log(3)} \quad$ and $\quad D(f) = D(f') = \mathbb{R}$, as $\cosh(x) \geq 1$ for all x.

6. Calculate the following limit $\lim_{x\to 2} \dfrac{\log(x-1)}{x-2}$.

Solution: Let $f(x) = \log(x-1)$ and $g(x) = x-2$. Then we have $\lim\limits_{x\to 2} f(x) = 0$, $\lim\limits_{x\to 2} g(x) = 0$ and $g'(x) = 1 \neq 0$. So the hypothesis of the L'Hôpital rule is satisfied, so we have,

$$\lim_{x\to 2} \frac{\log(x-1)}{x-2} \overset{\text{L'Hôpital}}{=} \lim_{x\to 2} \frac{\frac{1}{x-1}}{1} = 1.$$

7. Let the function $f :]-\pi/2, \pi/2[\to \mathbb{R}$ be defined by $f(x) = \log(1+\sin(x))$. What is the Taylor expansion of order 3 of f at $x = 0$.

(a) $x - \frac{x^3}{6} + \cdots$

(b) $x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$

(c) $x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

(d) $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

Solution: (c) is correct. We use the following Taylor expansions

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\varepsilon(x)$$

$$\sin(x) = x - \frac{x^3}{6} + x^3\varepsilon(x)$$

So

$$\log(1+\sin(x)) = (x - \frac{x^3}{6}) - \frac{1}{2}(x)^2 + \frac{1}{3}(x)^3 + x^3\varepsilon(x) = x - \frac{x^2}{2} + \frac{x^3}{6} + x^3\varepsilon(x)$$

8. Find the Taylor expansion of order n at $x = 0$ of the following functions.

(a) $f(x) = e^{\sin(x)}, \quad n = 4$

(b) $f(x) = \sqrt{1+\sin(x)}, \quad n = 3$

Solution:

(a) We use the 3rd order Taylor expansion around the point $x = 0$ of the exponential function which is valid for all $x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + x^4\varepsilon(x) \qquad \text{and} \qquad \sin(x) = x - \frac{x^3}{3!} + x^4\varepsilon(x).$$

So

$$\begin{aligned} e^{\sin(x)} &= 1 + \left(x - \frac{x^3}{6} + x^4\varepsilon(x)\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + x^4\varepsilon(x)\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + x^4\varepsilon(x)\right)^3 \\ &\quad + \frac{1}{24}\left(x - \frac{x^3}{6} + x^4\varepsilon(x)\right)^4 + x^4\varepsilon(x) \\ &= 1 + \left(x - \frac{x^3}{6}\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{1}{6}x^3 + \frac{1}{24}x^4 + x^4\varepsilon(x) \\ &= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + x^4\varepsilon(x)\,. \end{aligned}$$

(b) The 3rd order Taylor expansion around the point $x = 0$ of $\sin(x)$ and $(1+y)^{1/2}$ are

$$\sin(x) = x - \frac{x^3}{6} + x^3\varepsilon(x) \qquad \text{and} \qquad \sqrt{1+y} = 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^3}{16} + y^3\varepsilon(y)\,.$$

where by substituting $y = \sin(x)$, we get

$$\begin{aligned} \sqrt{1+\sin(x)} &= 1 + \frac{1}{2}\left(x - \frac{x^3}{6} + x^3\varepsilon(x)\right) - \frac{1}{8}\left(x - \frac{x^3}{6} + x^3\varepsilon(x)\right)^2 \\ &\quad + \frac{1}{16}\left(x - \frac{x^3}{6} + x^3\varepsilon(x)\right)^3 + x^3\varepsilon(x) \\ &= 1 + \frac{1}{2}\left(x - \frac{x^3}{6}\right) - \frac{1}{8}x^2 + \frac{1}{16}x^3 + x^3\varepsilon(x) \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + x^3\varepsilon(x)\,. \end{aligned}$$

9. For each one of the following functions, determine whether the funciton is differentiable at $x = 0$. If yes, also compute the derivative at $x = 0$:

(a) $f(x) = \begin{cases} x+1, & x \geq 0 \\ x, & x < 0 \end{cases}$;

(b) $f(x) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}$;

(c) $f(x) = \begin{cases} \frac{\sin(x)-x}{x}, & x > 0 \\ 0, & x = 0 \\ \frac{\cos(x)-\frac{x^2}{2}}{x^4}, & x < 0 \end{cases}$.

Solution:

(a) The function is not continuous at 0, as the limit from the right is 1 and the limit from the left is 0. As a differentiable function is continuous, then f is not differentiable at 0.

(b) We claim that the function is differentiable at 0 with derivative $f'(0) = 0$. Indeed, we compute

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^2 - 0}{h} = \lim_{h\to 0^+} h = 0.$$

Similarly, we have

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{h^3 - 0}{h} = \lim_{h\to 0^-} h^2 = 0.$$

Since the left-hand limit and the right-hand limit exist and agree, we conclude that

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = 0.$$

Thus, f is differentiable at 0 with $f'(0) = 0$.

(c) We claim that the function is not continuous at 0. Then, if that is the case, we can conclude as in case (a). One can see that $\lim_{x\to 0^+} f(x) = 0 = f(x)$. Yet, it suffices to show that $\lim_{x\to 0^-} f(x) \neq 0 = f(0)$. To this end, we compute the limit

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\cos(x) - \frac{x^2}{2}}{x^4} = +\infty$$

as the limit of the numerator is 1, while the denominator goes to 0 from above (hence the sign + for ∞).

10. Find the vertical and horizontal asymptotes of the function $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

Solution: A vertical asymptotes cannot exist in a point where the function is defined, so here potentially $x = 0$ is a candidate for a vertical asymptote. In fact, we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = +\infty \qquad \text{and} \qquad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

So f has a vertical asymptote at $x = 0$.

A horizontal asymptote (if it exists) is characterized by the limit of the function f at infinity ($\pm\infty$). Here we have

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{1}{x} = 0 \qquad \text{et} \qquad \lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{1}{x} = 0$$

So indeed f has an horizontal asymptote at $y = 0$.

11. State if the following statements are true or false. Let $f, g : I \to \mathbb{R}$ be two convex functions, where $I \subset \mathbb{R}$ is some interval. If it is true, prove it. If not, give a counter example.

    (a) The function $f + g$ is convex.

(b) The function $h = f \cdot g$ is convex.

(c) If g is increasing then the function $h = g \circ f$ is convex.

Solution:

(a) True. Using the definition we have

$$\begin{aligned}(f+g)(\lambda a+(1-\lambda)b) &= f(\lambda a+(1-\lambda)b)+g(\lambda a+(1-\lambda)b)\\ &\leq \lambda f(a)+(1-\lambda)f(b)+\lambda g(a)+(1-\lambda)g(b)\\ &= \lambda(f+g)(a)+(1-\lambda)(f+g)(b)\end{aligned}$$

So $f+g$ is convex.

(b) False. Take for example $f(x) = -\sqrt{x}$ and $g(x) = 1/x$ on $I =]0, +\infty[$. Then we have $h = f \cdot g = -\frac{1}{\sqrt{x}}$ which is a concave function. So in general if f and g are convex functions, we cannot say much about $f \cdot g$.

(c) True. Note that if f is convex then $f(\lambda a + (1 - \lambda b)) \leq \lambda f(a) + (1 - \lambda)f(b)$ now take $x_1 = f(\lambda a + (1 - \lambda b))$ and $x_2 = \lambda f(a) + (1 - \lambda)f(b)$. Since g is increasing and $x_1 \leq x_2$ then $g(x_1) \leq g(x_2)$. Finally, using convexity of g we can write:

$$g(x_1) \leq g(x_2) \Rightarrow g(f(\lambda a+(1-\lambda b))) \leq g(\lambda f(a)+(1-\lambda)f(b)) \leq \lambda g(f(a))+(1-\lambda)g(f(b))$$

Or we can write:

$$h(\lambda a + (1-\lambda)b) \leq \lambda h(a) + (1-\lambda)h(b)$$

so h is convex.

12. Consider $f\colon]a, b[\mapsto \mathbb{R}$. Let $g\colon]c, d[\mapsto \mathbb{R}$ be the restriction to f to the interval $]c, d[\subset]a, b[$, i.e., $f(x) = g(x) \quad \forall x \in]c, d[$. Show that

(a) If $f \in C^n(]a, b[, \mathbb{R})$ then $g \in C^n(]c, d[, \mathbb{R})$.

(b) If f is Lipschitz continuous, then g is Lipschitz continuous.

Solution:

(a) Note that $\forall x \in]c, d[$ we have $g^{(k)}(x) = f^{(k)}(x) \quad \forall k = 1, \cdots, n$ which shows $g \in \mathbb{C}^n(c, d)$.

(b) Since f is Lipschitz continuous with constant L we have $\forall x, y \in]c, d[, \quad |g(x) - g(y)| = |f(x) - f(y)| \leq L|x - y|$. Hence, g is Lipschitz continuous.

13. Find the local extrema and the absolute maximum and minimum of $f(x) = x^2 - \left|x + \frac{1}{4}\right| + 1$ in $[-1, 1]$.

Solution: Before calculating the derivatives, we rewrite f by distinguishing two cases. we have

$$f(x) = \begin{cases} x^2 + x + \frac{5}{4}\,, & -1 \leq x \leq -\frac{1}{4} \\ x^2 - x + \frac{3}{4}\,, & -\frac{1}{4} < x \leq 1 \end{cases}, \qquad f'(x) = \begin{cases} 2x + 1\,, & -1 < x < -\frac{1}{4} \\ 2x - 1\,, & -\frac{1}{4} < x < 1 \end{cases}$$

For $x_0 = -\frac{1}{4}$ we have

$$f'_r(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to -\frac{1}{4}^+} \frac{x^2 - x - \frac{5}{16}}{x + \frac{1}{4}} = \lim_{x \to -\frac{1}{4}^+} \frac{\left(x - \frac{5}{4}\right)\left(x + \frac{1}{4}\right)}{x + \frac{1}{4}} = -\frac{3}{2}$$

$$f'_l(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to -\frac{1}{4}^-} \frac{x^2 + x + \frac{3}{16}}{x + \frac{1}{4}} = \lim_{x \to -\frac{1}{4}^-} \frac{\left(x + \frac{3}{4}\right)\left(x + \frac{1}{4}\right)}{x + \frac{1}{4}} = \frac{1}{2}$$

and so f is not differentiable in this point. Also $f''(x) = 2$ For all $x \in \left]-1, -\frac{1}{4}\right[\cup \left]-\frac{1}{4}, 1\right[$. So local and absolute extrema are listed bellow:

(a) Stationary points: $f'(x) = 0 \Rightarrow x_1 = -\frac{1}{2}$ or $x_2 = \frac{1}{2}$. Since $f''(x_1) = f''(x_2) > 0$, x_1 and x_2 are local minimums. We have $f(x_1) = 1$ and $f(x_2) = \frac{1}{2}$.

(b) Points where f' does not exist: The only point is $x_0 = -\frac{1}{4}$ since we have $f'_r(x_0) = -\frac{3}{2}$ and $f'_l(x_0) = \frac{1}{2}$. Looking at the sign of the derivative at x_0 we deduce that this point is a local maximum. We have $f(x_0) = \frac{17}{16}$.

(c) Boundaries of the domain: Since f is continuous on $[-1, 1]$, we look at the sign of f' in the boundaries of $[-1, 1]$ and notice that f has local maximums at $a = -1$ and $b = 1$. We have $f(a) = \frac{5}{4}$ and $f(b) = \frac{3}{4}$.

$$(a), (b), (c) \quad \Rightarrow \quad \begin{cases} \text{global maximum at} & x = -1, \quad f(-1) = \frac{5}{4} \\ \text{global minimum at} & x = \frac{1}{2}, \quad f\left(\frac{1}{2}\right) = \frac{1}{2} \end{cases}$$

14. Let $a, b \in \overline{\mathbb{R}}$, $a < b$. Let $f :]a, b[\to \mathbb{R}$ be a differentiable function. State if the following statements are true or false. If it is true, prove it. If not, give a counter example.
    (a) If f' is bounded, then f is Lipschitz continuous with Lipschitz contstant $k = \sup_{x \in]a,b[} |f'(x)|$.
    (b) If f is Lipschitz continuous, then it is uniformly continuous.
    (c) If f' is bounded then f is uniformly continuous.

Solution:

(a) True. We use the Mean Value Theorem as follows. For every $x < y$ in $]a, b[$ there exists $c \in [x, y]$ such that $f(y) - f(x) = f'(c)(y - x)$. Hence

$$|f(y) - f(x)| = |f'(c)||y - x| \leq k|y - x|.$$

(b) True. Let $k \geq 0$ such that f is Lipschitz continuous with constant k. Let $\varepsilon > 0$ and $x, y \in]a, b[$. We compute that

$$|f(x) - f(y)| \leq k|x - y| \leq \varepsilon$$

holds if $|x - y| \leq \frac{\varepsilon}{k}$. So f is uniformly continuous with $\delta := \frac{\varepsilon}{k}$.

(c) True. Combine the previous statements.

15. Study the function $f(x) = \dfrac{x}{x^2 - 1}$ and sketch its graph (domain, range, symmetries, roots, continuity, differentiability, stationary points, extrema, convexity, inflection points, asymptotes).

Solution:

(a) $D(f) = \mathbb{R} \setminus \{-1, 1\}, \quad \mathrm{Im}(f) = \mathbb{R}$

(b) Odd, non-periodic

(c) $f(x) = 0 \Leftrightarrow x = 0$

(d) f is continuous since it is composition of continuous functions on $D(f)$.

(e) f is differentiable on $D(f)$

$$f'(x) = -\frac{x^2+1}{\left(x^2-1\right)^2}\,, \quad D(f') = D(f) \qquad \text{and} \qquad f''(x) = \frac{2x\left(x^2+3\right)}{\left(x^2-1\right)^3}\,, \quad D(f'') = D(f)$$

(f)
- $f'(x) < 0$ for all $x \in D(f')$, so there are no stationary points.
- $f''(x) = 0 \Leftrightarrow x = 0$. We also calculate f''':

  $$f'''(x) = -\frac{6(x^4+6x^2+1)}{(x^2-1)^4} < 0 \quad \text{for all } x \in D(f).$$

  Since $f'''(0) = -6 \neq 0$, f has an inflection point in $x = 0$.

(g)
- Monotonicity:

x	$]-\infty,-1[$	$]-1,1[$	$]1,\infty[$
f'	< 0	< 0	< 0
f	decreasing	decreasing	decreasing

  Note that f is strictly decreasing in the intervals in the table but not on the entire domain $D(f)$.
- Convexity and concavity:

x	$]-\infty,-1[$	$]-1,0[$	$]0,1[$	$]1,\infty[$
f''	< 0	> 0	< 0	> 0
f	concave	convex	concave	convex

(h)
- Vertical asymptotes: f is not defined in $x = \pm 1$ and we have $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to 1^+} f(x) = \infty$ also $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to 1^-} f(x) = -\infty$, so there of vertical asymptotes at $x = \pm 1$.
- horizontal asymptotes: $\lim\limits_{x\to\pm\infty} f(x) = 0$, so one horizontal asymptote at $y = 0$.

16. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = e^x$. Compute the upper and lower Darboux sums for the regular partitions σ_n. Is f integrable?

Solution: The regular partition σ_n of the interval $[0,1]$ is $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1$. Since the function e^x is increasing, we have

$$\underline{S}_{\sigma_n} = \sum_{i=1}^{n}\left(\inf_{x\in[\frac{i-1}{n},\frac{i}{n}]} f(x)\right)\left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum_{i=1}^{n} e^{\frac{i-1}{n}}\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}(e^{\frac{1}{n}})^{i-1} = \frac{1}{n}\frac{e-1}{e^{\frac{1}{n}}-1},$$

$$\overline{S}_{\sigma_n} = \sum_{i=1}^{n}\left(\sup_{x\in[\frac{i-1}{n},\frac{i}{n}]} f(x)\right)\left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum_{i=1}^{n} e^{\frac{i}{n}}\frac{1}{n} = \frac{e^{\frac{1}{n}}}{n}\sum_{i=1}^{n}(e^{\frac{1}{n}})^{i-1} = \frac{e^{\frac{1}{n}}}{n}\frac{e-1}{e^{\frac{1}{n}}-1}.$$

We want to compute the limits $\lim_{n\to+\infty} \underline{S}_{\sigma_n}$ and $\lim_{n\to+\infty} \overline{S}_{\sigma_n}$, but we compute first $\lim_{n\to+\infty} n(e^{\frac{1}{n}}-1)$. We observe that since the function f is differentiable, we have

$$\lim_{n\to+\infty} n(e^{\frac{1}{n}}-1) = \lim_{n\to+\infty} \frac{f(\frac{1}{n})-f(0)}{\frac{1}{n}-0} = f'(0) = 1$$

Then

$$\begin{aligned}
\lim_{n\to+\infty} \underline{S}_{\sigma_n} &= \lim_{n\to+\infty} \frac{1}{n}\frac{e-1}{e^{\frac{1}{n}}-1} = (e-1)(\lim_{n\to+\infty} n(e^{\frac{1}{n}}-1))^{-1} = e-1 \\
\lim_{n\to+\infty} \overline{S}_{\sigma_n} &= \lim_{n\to+\infty} \frac{e^{\frac{1}{n}}}{n}\frac{e-1}{e^{\frac{1}{n}}-1} = (e-1)(\lim_{n\to+\infty} e^{\frac{1}{n}})(\lim_{n\to+\infty} n(e^{\frac{1}{n}}-1))^{-1} = e-1
\end{aligned}$$

Since

$$\lim_{n\to+\infty} \underline{S}_{\sigma_n} \le \underline{S} \le \overline{S} \le \lim_{n\to+\infty} \overline{S}_{\sigma_n}$$

we obtain that $\underline{S} = \overline{S}$ and the function f is integrable.

17. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f\colon \mathbb{R}\to\mathbb{R}$ be a continuous function on $[a,b]\subset D(f)$, $a<b$, and differentiable on $]a,b[$.

    (a) If $f'(x)\ge 0$ for all $x\in\,]a,b[$, then f is increasing on $[a,b]$.

    (b) If f is increasing on $[a,b]$, then $f'(x)\ge 0$ for all $x\in\,]a,b[$.

    (c) If f is strictly increasing on $[a,b]$, then $f'(x)>0$ for all $x\in\,]a,b[$.

    (d) If $f'(x)>0$ for all $x\in\,]a,b[$, then f is strictly increasing on $[a,b]$.

    (e) If $\lim\limits_{x\to a^+} f'(x)=\ell$ exists, then f is differentiable from right at a and the right derivative is $f'_d(a)=\ell$.

Solution:

(a) True. Direct consequence of $f'(x)\ge 0$ and MVT.

(b) True. For all $x\in\,]a,b[$, the derivative of f is defined by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Since f is increasing on $[a,b]$, $f(x+h)-f(x)$ has the same sign as h. So the quotient inside the limit is always positive so $f'(x)\ge 0$.

(c) False. Take for example $f\colon[-1,1]\to\mathbb{R}$ defined by $f(x)=x^3$. This function is strictly increasing on $[-1,1]$ but $f'(0)=0$.

(d) True. Direct consequence of $f'(x)>0$ and MVT.

(e) True. We want to compute $f'_d(a)$, which by definition is

$$\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

By the mean value theorem, for every $x \in]a, b[$ there exists $c = c(x) \in]a, x[$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(c(x))$$

(Let us notice that $c(x)$ is a function of x, a fancy function but still a function; and let us stress again that $a \leq c(x) \leq x$).

To conclude the exercise we just have to show that

$$\lim_{x \to a^+} f'(x) = \lim_{x \to a^+} f'(c(x)),$$

This follows from the definition of limit and the fact that $a \leq c(x) \leq x$. Indeed, we have to show that for every ϵ ther exists a δ such that if $|a - x| < \delta$ then $|\ell - f(c(x))| \leq \epsilon$. We know that there exists δ such that if $|a - x| < \delta$ then $|\ell - f(x)| \leq \epsilon$, but for this very same δ we also have that $|a - c(x)| \leq \delta$, so $|\ell - f(c(x))| \leq \epsilon$, and this prove the claim.

18. Using the definition of convex functions, show that the function $f(x) = x^2$ is convex.

Solution: We must show that for any $\lambda \in [0, 1]$ and $a, b \in D_f = \mathbb{R}$ we have

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda) f(b)$$

for convex functions and

$$f(\lambda a + (1 - \lambda)b) \geq \lambda f(a) + (1 - \lambda) f(b)$$

for concave functions.

We have

$$\begin{aligned}
&f(\lambda a + (1 - \lambda)b) - (\lambda f(a) + (1 - \lambda) f(b)) \\
&= \lambda^2 a^2 + (1 - \lambda)^2 b^2 + 2\lambda(1 - \lambda)ab - \lambda a^2 - (1 - \lambda)b^2 \\
&= -\lambda(1 - \lambda)a^2 - \lambda(1 - \lambda)b^2 + 2\lambda(1 - \lambda)ab \\
&= -\lambda(1 - \lambda)(a - b)^2 \leq 0
\end{aligned}$$

The last equality is true for all $\lambda \in [0, 1]$. So the function is convex.

19. Find the local extrema and the absolute maximum and minimum of $f(x) = (x-1)^2 - 2\,|2 - x|$ in $]2, 3[$

Solution: Since $2 - x < 0$ for all $x \in]2, 3[=: I$, there is no need for distinguishing two cases for f. We have

$$f(x) = (x - 1)^2 + 2(2 - x) = x^2 - 4x + 5 \quad \text{and} \quad f'(x) = 2(x - 2) \quad \text{for all } x \in I$$

The local and global extrema are listed below:

(a) Stationary points: $f'(x) \neq 0$ for all $x \in I$, so no stationary points.

(b) Points where f' does not exist: f' exists on I.

(c) Boundaries of the domain: The domain I is an open interval and so f does not take any extrema.

So the function f does not have any local or global extrema on I

20. Study the function $f(x) = \dfrac{3x^2 - x}{2x - 1}$ and sketch its graph (domain, range, symmetries, roots, continuity, differentiability, stationary points, extrema, convexity, inflection points, asymptotes).

Solution:

(a) $D(f) = \mathbb{R} \setminus \frac{1}{2}, \quad \operatorname{Im}(f) = \mathbb{R} \setminus \left]1 - \frac{\sqrt{3}}{2}, 1 + \frac{\sqrt{3}}{2}\right[$

(b) not even, not odd and not periodic.

(c) $f(x) = 0 \quad \Leftrightarrow \quad 3x^2 - x = 0 \quad \Leftrightarrow \quad x = 0$ or $x = \frac{1}{3}$.

(d) Continuous on $D(f)$ (composition of continuous functions)

(e) f is differentiable on $D(f)$

$$f'(x) = \frac{(6x-1)(2x-1) - 2(3x^2 - x)}{(2x-1)^2} = \frac{6x^2 - 6x + 1}{(2x-1)^2}\,, \qquad D(f') = D(f)$$

$$\begin{aligned} f''(x) &= \frac{(12x-6)(2x-1)^2 - 4(6x^2 - 6x + 1)(2x-1)}{(2x-1)^4} \\ &= \frac{(12x-6)(2x-1) - 4(6x^2 - 6x + 1)}{(2x-1)^3} = \frac{2}{(2x-1)^3}\,, \qquad D(f'') = D(f) \end{aligned}$$

(f)
- $f'(x) = 0 \quad \Leftrightarrow \quad 6x^2 - 6x + 1 = 0 \quad \Leftrightarrow \quad x = \frac{6 \pm \sqrt{36-24}}{12} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$
  Sp f has a stationary point at $x_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}$ and $x_2 = \frac{1}{2} - \frac{\sqrt{3}}{6}$. Since

  $$f''(x_{1,2}) = \frac{2}{\left(2\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right) - 1\right)^3} = \frac{2}{\left(\pm\frac{\sqrt{3}}{3}\right)^3} = \pm\frac{2}{3^{-3/2}} = \pm 6\sqrt{3}\,,$$

  It follows that x_1 is a local minimum (since $f''(x_1) > 0$) and x_2 a local maximum (since $f''(x_2) < 0$) of f.
- Since $f''(x) \neq 0$ for all $x \in D(f)$, f does not have an inflection point.

(g)
- Monotonicity:

x	$\left]-\infty, \frac{1}{2} - \frac{\sqrt{3}}{6}\right[$	$\left]\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2}\right[$	$\left]\frac{1}{2}, \frac{1}{2} + \frac{\sqrt{3}}{6}\right[$	$\left]\frac{1}{2} + \frac{\sqrt{3}}{6}, \infty\right[$
f'	> 0	< 0	< 0	> 0
f	increasing	decreasing	decreasing	increasing

- Convexity and concavity:

x	$\left]-\infty, \frac{1}{2}\right[$	$\left]\frac{1}{2}, \infty\right[$
f''	< 0	> 0
f	concave	convex

(h)
- Vertical asymptotes: f is not defined in $x = \frac{1}{2}$ and

$$\lim_{x \to \frac{1}{2}^{\pm}} f(x) = \lim_{x \to \frac{1}{2}^{\pm}} \frac{x(3x-1)}{2x-1} = \pm\infty$$

  Since $x(3x-1) > 0$ for x close to $\frac{1}{2}$. So f has a vertical asymptote at $x = \frac{1}{2}$.

- Horizontal asymptotes:

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x(3x-1)}{2x-1} = \lim_{x \to \pm\infty} \frac{3x-1}{2-\frac{1}{x}} = \pm\infty,$$

  So f does not have any horizontal asymptotes.

- obliqued asymptotes:

$$a = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{3x-1}{2x-1} = \frac{3}{2} \qquad \text{et}$$

$$b = \lim_{x \to \pm\infty} \big(f(x) - ax\big) = \lim_{x \to \pm\infty} \left(\frac{x(2x-1+x)}{2x-1} - \frac{3}{2}x\right) = \lim_{x \to \pm\infty} \left(-\frac{x}{2} + \frac{x^2}{2x-1}\right)$$
$$= \lim_{x \to \pm\infty} \frac{x}{2(2x-1)} = \frac{1}{4}$$

  So f has an obliqued asymptotes with equation $y = ax + b = \frac{3}{2}x + \frac{1}{4}$.

(i) We find the value of f at $x_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$:

$$f(x_{1,2}) = \frac{\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right)\left(3\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right) - 1\right)}{\pm 3^{-1/2}} = \pm\sqrt{3}\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right)\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)$$
$$= \left(\pm\frac{\sqrt{3}}{2} + \frac{1}{2}\right)\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)^2 = 1 \pm \frac{\sqrt{3}}{2},$$

Since $f(x_1) > f(x_2)$ and knowing the nature of the local extrema of f in x_1 and x_2 that $\operatorname{Im}(f) = \mathbb{R} \setminus \left]1 - \frac{\sqrt{3}}{2}, 1 + \frac{\sqrt{3}}{2}\right[$.

21. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathbb{R}$ with $g'(x) \neq 0$ for all $x \in \mathbb{R}$.

    (a) If $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} g(x) = \infty$, then $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \frac{f'(x)}{g'(x)}$.

    (b) If $\lim\limits_{x \to \infty} \frac{f'(x)}{g'(x)}$ does not exist, then $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)}$ does not exist.

Solution:

(a) False. Take for example $f(x) = x + \sin(x)$ and $g(x) = x$. In this example we have $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \left(1 + \frac{\sin(x)}{x}\right) = 1$ but $\frac{f'(x)}{g'(x)} = 1 + \cos(x)$ does not have a limit (One of the hypotheses of L'Hôpital rule is not satisfied).

(b) False. Take the same functions of the previous exercise.

22. Using the definition of convex functions, show that the function $f(x) = \frac{1}{x}, \quad x \in]0, +\infty[$ is convex.

Solution: We have

$$\begin{aligned}
&f(\lambda a + (1-\lambda)b) - (\lambda f(a) + (1-\lambda)f(b)) \\
&= \frac{1}{\lambda a + (1-\lambda)b} - \frac{\lambda}{a} - \frac{1-\lambda}{b} \\
&= \frac{ab - b\lambda(\lambda a + (1-\lambda)b) - a(1-\lambda)(\lambda a + (1-\lambda)b)}{(\lambda a + (1-\lambda)b)ab} \\
&= \frac{ab - \lambda^2 ab - \lambda(1-\lambda)b^2 - \lambda(1-\lambda)a^2 - (1-\lambda)^2 ab}{(\lambda a + (1-\lambda)b)ab} \\
&= \frac{(1 - \lambda^2 - (1-\lambda)^2)ab - \lambda(1-\lambda)b^2 - \lambda(1-\lambda)a^2}{(\lambda a + (1-\lambda)b)ab} \\
&\leq \frac{(1 - \lambda^2 + 2\lambda - (1-\lambda)^2)ab - \lambda(1-\lambda)b^2 - \lambda(1-\lambda)a^2}{(\lambda a + (1-\lambda)b)ab} \\
&= \frac{((1-\lambda)^2 - (1-\lambda)^2)ab - \lambda(1-\lambda)b^2 - \lambda(1-\lambda)a^2}{(\lambda a + (1-\lambda)b)ab} \\
&= \frac{-\lambda(1-\lambda)b^2 - \lambda(1-\lambda)a^2}{(\lambda a + (1-\lambda)b)ab} \leq 0
\end{aligned}$$

The last inequality holds since the denominator is always positive and the numerator is strictly negative. So the function is convex.