Analysis 1 - Exercise Set 11

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Let $a, b \in \mathbb{Z}$, $b > 0$. Show that $\sqrt[b]{x^a} = e^{\frac{a}{b}\log(x)}$ for all real numbers $x > 0$.

   (b) Compute the derivative of the following functions:

   (i) $f(x) = x^a : \mathbb{R}_+^* \to \mathbb{R}$, $a \in \mathbb{R}$; show that f is strictly increasing when $a > 0$ and strictly decreasing when $a < 0$;

   (ii) $f(x) = a^x : \mathbb{R} \to \mathbb{R}_+^*$, $a \in \mathbb{R}_+^*$; show that f is strictly increasing when $a > 1$ and strictly decreasing when $a < 1$;

   (iii) $f(x) = \log_a(x) : \mathbb{R}_+^* \to \mathbb{R}$, $a \in \mathbb{R}_+^*$; show that f is strictly increasing when $a > 1$ and strictly decreasing when $a < 1$;

Solution:

(a) $\sqrt[b]{x^a}$ is the unique positive number such that $(\sqrt[b]{x^a})^b = x^a$. The number $e^{\frac{a}{b}\log(x)}$ is positive, because $e^y > 0$ for all $y \in \mathbb{R}$. So it suffices to check that

$$(e^{\frac{a}{b}\log(x)})^b = e^{a\log(x)} = (e^{\log(x)})^a = x^a.$$

(b) (i) $(x^a)' = (e^{a\log(x)})' = \frac{a}{x}e^{a\log(x)} = ax^{a-1}$. From this we see that, since $x > 0$, $(x^a)' > 0$ if $a > 0$ and hence x^a is strictly increasing. Likewise, $(x^a)' < 0$ when $a < 0$ and hence x^a is strictly decreasing.

(ii) $(a^x)' = (e^{x\log(a)})' = \log(a)e^{x\log(a)} = \log(a)a^x$. Since $\log(a) > 0$ for $a > 1$ and $\log(a) < 0$ for $0 < a < 1$, we see that $(a^x)' > 0$ if $a > 1$, and $(a^x)' < 0$ if $a < 1$. Therefore, a^x is strictly increasing if $a > 1$ and strictly decreasing if $a < 1$.

(iii) $(\log_a(x))' = \left(\frac{\log(x)}{\log(a)}\right)' = \frac{1}{x\log(a)}$. The monotonicity results follows from that $\log_a(x)$ is the inverse of a^x and that the inverse of a strictly increasing/decreasing function is strictly increasing/decreasing.

2. For a complex number of the form e^{ix}, $x \in \mathbb{R}$, we defined

$$\cosh(ix) := \frac{e^{ix} + e^{-ix}}{2}, \quad \sinh(ix) := \frac{e^{ix} - e^{-ix}}{2}.$$

   (a) Compute the complex numbers $\cosh(ix)$, $\sinh(ix)$;

   (b) For each of the functions $\cosh(x), \sinh(x), \tanh(x), \coth(x)$ compute the derivative and the domain of the derivative.
   Which of these functions are invertible on the domain $\mathbb{R}$? which on $\mathbb{R}_+^*$? Recall that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \ \sinh(x) = \frac{e^x - e^{-x}}{2}, \ \tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \ \coth(x) = \frac{1}{\tanh(x)}.$$

(c) Compute the derivatives of the inverses of the functions in (b) and their domains.

Solution:

(a)

$$\cosh(ix) = \frac{e^{ix}+e^{-ix}}{2} = \frac{1}{2}(\cos(x)+i\sin(x)+\cos(-x)+i\sin(-x)) = \cos(x)$$

$$\sinh(ix) = \frac{e^{ix}-e^{-ix}}{2} = \frac{1}{2}(\cos(x)+i\sin(x)-(\cos(-x)+i\sin(-x))) = i\sin(x)$$

(b) $D(\cosh(x)) = D(\sinh(x)) = \mathbb{R}$. Since $\cosh(x) > 0$ for all $x \in \mathbb{R}$, then $D(\tanh(x)) = \mathbb{R}$. Since $\sinh(x) = 0$ only if $x = 0$, we have $D(\coth(x)) = \mathbb{R} \setminus \{0\}$.

Using the fact that $(e^{ax})' = ae^{ax}$ for all $a \in \mathbb{R}$ we see immediately that

$$\cosh(x)' = \frac{e^x - e^{-x}}{2} = \sinh(x), \quad \sinh(x)' = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

and $D(\cosh(x)') = D(\sinh(x)') = \mathbb{R}$. Since $\sinh(x)' > 0$ for all $x \in \mathbb{R}$, then the function $\sinh(x)$ is strictly increasing and hence invertible on the domain $\mathbb{R}$. We compute $\lim_{x\to+\infty}\sinh(x) = +\infty$ and $\lim_{x\to-\infty}\sinh(x) = -\infty$, so $R(\sinh(x)) = \mathbb{R}$ because $\sinh(x)$ is a continuous function. So $R(\sinh(x)) = \mathbb{R}$. We compute $\lim_{x\to\pm\infty}\cosh(x) = +\infty$, so the function $\cosh(x)$ is not invertible on the domain $\mathbb{R}$, but $\cosh(x)' > 0$ for all $x > 0$, so $\cosh(x)$ is strictly increasing and hence invertible on the domain $\mathbb{R}^*_+$. Also, we observe that $\cosh(x)$ has a minimum in $x = 0$, because it is stricly decreasing on $]-\infty, 0[$ and strictly increasing on $]0, +\infty[$. So $R(\cosh(x)) = [\cosh(0), +\infty[= [1, +\infty[$.

Using the rule of derivative of a quotient, we see that

$$\tanh(x)' = \frac{\sinh(x)'\cosh(x) - \sinh(x)\cosh(x)'}{\cosh(x)^2} = \frac{\cosh(x)^2 - \sinh(x)^2}{\cosh(x)^2} = \frac{1}{\cosh(x)^2}$$

$$\coth(x)' = \frac{\cosh(x)'\sinh(x) - \cosh(x)\sinh(x)'}{\sinh(x)^2} = \frac{\sinh(x)^2 - \cosh(x)^2}{\sinh(x)^2} = -\frac{1}{\sinh(x)^2}$$

Then $D(\tanh(x)') = \mathbb{R}$, $D(\coth(x)') = \mathbb{R} \setminus \{0\}$. Since $\tanh(x)' > 0$ for all $x \in \mathbb{R}$ the function $\tanh(x)$ is strictly increasing, and hence invertible. Since $\coth(x)' < 0$ for all $x \neq 0$, the functions $\coth(x)|_{]-\infty,0[}, \coth(x)|_{]0,+\infty[}$ are both strictly decreasing and hence invertible on their domains. To compute the range of monotone functions defined on intervals, it suffices to compute the limits at the extremities of the domain. We have

$$\lim_{x\to-\infty}\tanh(x) = \lim_{x\to-\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = -1, \quad \lim_{x\to+\infty}\tanh(x) = \lim_{x\to+\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = 1$$

so $R(\tanh(x)) = [-1, 1]$.

$$\lim_{x\to-\infty}\coth(x) = \lim_{x\to-\infty}\frac{e^x + e^{-x}}{e^x - e^{-x}} = -1, \quad \lim_{x\to+\infty}\coth(x) = \lim_{x\to+\infty}\frac{e^x + e^{-x}}{e^x - e^{-x}} = 1,$$

$$\lim_{x\to0^-}\coth(x) = \lim_{x\to0^-}\frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x\to0^-}\frac{1+e^{-2x}}{1-e^{-2x}} = -\infty,$$

$$\lim_{x\to0^+}\coth(x) = \lim_{x\to0^+}\frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x\to0^+}\frac{1+e^{-2x}}{1-e^{-2x}} = +\infty$$

so $R(\coth(x)) =]-\infty, -1[\cup]1, +\infty[$.

(c) Note that we know the ranges of the hyperbolic functions above, which will be the domains for the inverses. Therefore, we automatically get the domains of the inverses from the ranges in (b).

Consider $y = \cosh(x)$. Then, if $x \geq 0$, we have $\cosh^{-1}(y) = x$. By the rule for the derivative of the inverse function, we have $(\cosh^{-1}(y))' = \frac{1}{(\cosh(\cosh^{-1}(y)))'}$. Thus, we have

$$
\begin{aligned}
(\cosh^{-1}(y))' &= \frac{1}{(\cosh(\cosh^{-1}(y)))'} \\
&= \frac{1}{\sinh(\cosh^{-1}(y))} \\
&= \frac{1}{\sqrt{\sinh^2(\cosh^{-1}(y))}} \\
&= \frac{1}{\sqrt{\cosh^2(\cosh^{-1}(y)) - 1}} \\
&= \frac{1}{\sqrt{y^2 - 1}},
\end{aligned}
$$

where in the second line we used that $\cosh' = \sinh$, in the third line we used that $\cosh^{-1}(y) = x \geq 0$ by assumption and that $\sinh(x) \geq 0$ when $x \geq 0$, which guarantees that $\sinh(x) = \sqrt{\sinh^2(x)}$ when $\sinh(x) \geq 0$. Lastly, in the fourth line we used the hyperbolic trigonometric identity $\cosh^2(t) - \sinh^2(t) = 1$. Recall that the domain of $\cosh^{-1}(y)$ is $[1, \infty)$; thus, the domain of its derivative is $(1, \infty)$.

We apply a similar argument. Notice that $y = \sinh(x)$ is invertible on $\mathbb{R}$, so there is no restriction to write $x = \sinh^{-1}(y)$. Then, we have

$$
\begin{aligned}
(\sinh^{-1}(y))' &= \frac{1}{(\sinh(\sinh^{-1}(y)))'} \\
&= \frac{1}{\cosh(\sinh^{-1}(y))} \\
&= \frac{1}{\sqrt{\cosh^2(\sinh^{-1}(y))}} \\
&= \frac{1}{\sqrt{\sinh^2(\sinh^{-1}(y)) + 1}} \\
&= \frac{1}{\sqrt{y^2 + 1}},
\end{aligned}
$$

where we argued as in the previous step and used that $\cosh(t) = \sqrt{\cosh^2(t)}$ for all t, as $\cosh(t) \geq 1$ for all t. Thus, the derivative is defined on all $\mathbb{R}$.

Applying a similar argument to tanh we write $y = \tanh(x)$, so $x = \tanh^{-1}(y)$, where $x \in \mathbb{R}$ and $y \in]-1, 1[$. Then, we have

$$\begin{aligned}
(\tanh^{-1}(y))' &= \frac{1}{(\tanh(\tanh^{-1}(y)))'} \\
&= \cosh^2(\tanh^{-1}(y)) \\
&= \frac{1}{1 - \tanh^2(\tanh^{-1}(y))} \\
&= \frac{1}{1 - y^2},
\end{aligned}$$

where we used the identity $\operatorname{sech}^2(t) = 1 - \tanh^2(t)$, which can be rearranged as $\cosh^2(t) = \frac{1}{1-\tanh^2(t)}$. So, the domain of the derivative coincides with the domain of $\tanh^{-1}(y)$, which is $]-1, 1[$.

Applying a similar argument to $y = \coth(x)$, we write

$$\begin{aligned}
(\coth^{-1}(y))' &= \frac{1}{(\coth(\coth^{-1}(y)))'} \\
&= -\sinh^2(\coth^{-1}(y)) \\
&= \frac{1}{1 - \coth^2(\coth^{-1}(y))} \\
&= -\frac{1}{1 - y^2} \\
&= \frac{1}{y^2 - 1},
\end{aligned}$$

where we used the identity $\operatorname{csch}^2(t) = \coth^2(t) - 1$, which can be rearranged as $\sinh^2(t) = \frac{1}{\coth^2(t)-1}$. So, the domain of the derivative coincides with the domain of $\coth^{-1}(y)$, which is $]-\infty, -1[\cup]1, +\infty[$.

3. For the following functions, find the stationary points and discuss whether they are points at which the function attains a local maximum or minimum.
   (a) $f(x) = x \log^2(x)$ in $]0, +\infty[$
   (b) $f(x) = 2\sin(x) + \cos(2x)$ in $[-\frac{1}{10}, \frac{1}{15}]$

Solution:

(a) The derivative is

$$f'(x) = \log(x)(\log(x) + 2)$$

and it is zero at $x = 1$ and $x = e^{-2}$. The derivative is positive in the interval $]0, e^{-2}[$, negative in $]e^{-2}, 1[$ and positive in $]1, +\infty[$, so e^{-2} is a local maximum and 1 is a local minimum. Since $f(1) = 0$ and $f(x) \geq 0$ for all x in the domain, it follows that the minimum at 1 is also global. On the other hand, the local maximum is not a global maximum, as $\lim_{x\to+\infty} f(x) = +\infty$.

(b) The derivative is

$$f'(x) = 2\cos(x) - 2\sin(2x) = 2\cos(x)(1 - 2\sin(2x)).$$

The derivative vanishes only if $\cos(x) = 0$ (i.e., $x = \frac{\pi}{2} + k\pi$) or $\sin(2x) = \frac{1}{2}$ (i.e., $x = \frac{\pi}{12} + k\pi$ or $x = \frac{5\pi}{12} + k\pi$), and no one of these values is in $\left[-\frac{1}{10}, \frac{1}{15}\right]$. So, f is strictly increasing on its domain and has no stationary points on its domain. Therefore, as the domain is a closed interval, we have a minimum at $-\frac{1}{10}$ and a maximum at $\frac{1}{15}$.

4. State if the following are true or false.

   (a) If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and has two roots, that is, there exist $x \neq y \in \mathbb{R},\ f(x) = 0 = f(y)$, then f' has at least one root.

   (b) The function $f(x) = \frac{\sin(x^2-2)}{e^{3x+1}+\sqrt{2x}}$ has a critical value in $]\sqrt{2}, \sqrt{2+\pi/2}[$.

Solution:

(a) True. If the two roots, a and b, are distinct, up to swapping them we may assume $a < b$. Then we apply Rolle's Theorem to the interval $[a, b]$ to prove that f' has a zero between them.

(b) True. We compute the derivative of $\frac{\sin(x^2-2)}{e^{3x+1}+\sqrt{2x}}$:

$$f'(x) = \frac{2x\cos(x^2-2)(e^{3x+1}+\sqrt{2x}) - \sin(x^2-2)(3e^{3x+1}+\sqrt{2x^{-1}}/2)}{(e^{3x+1}+\sqrt{2x})^2}.$$

We observe that the domain of f is $[0, +\infty[$; the above computation shows that f is differentiable on $]0, +\infty[$. Furthermore, f' is a continuous funciton on its domain $]0, +\infty[$. We compute

$$f'(\sqrt{2}) = \frac{2\sqrt{2}}{e^{3\sqrt{2}+1}+\sqrt{2\sqrt{2}}} > 0, \quad f'(\sqrt{2+\pi/2}) = -\frac{(3e^{3\sqrt{2+\pi/2}+1}+\sqrt{(4+\pi)^{-1}})}{e^{3\sqrt{2+\pi/2}+1}+\sqrt{2\sqrt{2+\pi/2}}} < 0$$

So by the intermediate value theorem applied to the continuous function f', there is $x_0 \in]\sqrt{2}, \sqrt{2+\pi/2}[$ such that $f'(x_0) = 0$.

5. State if the following are true or false.

   (a) If $f : E \to F$ is strictly increasing and bijective, then the inverse function is strictly increasing.

   (b) If $f(x) = x^2 - 2x$, then $(f \circ f)'(1) = 0$.

   (c) If a car traveled 210 km in 3 hours, then the speedometer must have read 70 km/h at least once.

Solution:

(a) True. If $a < b$ in F, then $f^{-1}(a) \neq f^{-1}(b)$ because f^{-1} is injective. Moreover, if $f^{-1}(a) > f^{-1}(b)$, then $a = f(f^{-1}(a)) > f(f^{-1}(b)) = b$ which is a contradiction.

(b) True. We have $f'(1) = 2 - 2 = 0$ and then $(f \circ f)'(1) = f'(f(1)) \cdot f'(1) = 0$.

(c) True. Let $f(t)$ be the traveled distance (Km) of the car at time t (h). Then $f'(t)$ is the speed of the car at time t. Now we apply the mean value theorem on the interval $[0, 3]$h. There should a time T such that

$$f'(T) = \frac{f(3) - f(0)}{3} = \frac{210}{3} = 70(\text{Km/h})$$

6. Find the inverse of the following functions if they exist. Give the domain of both functions.

(a) $f(x) = \left(\frac{1}{8}\right)^{1-x}$

(b) $f(x) = \log x - \log 2x + \log 3x$

Solution:

(a) The domain of f is all the real numbers $D_f = \mathbb{R}$. To find the inverse function we have:

$$y = \left(\frac{1}{8}\right)^{1-x} \Rightarrow \log y = (1 - x)\log(\frac{1}{8}) \Rightarrow x = 1 - \frac{\log y}{\log(\frac{1}{8})}$$

So the inverse function is given by $f^{-1}(x) = 1 - \frac{\log x}{\log(\frac{1}{8})}$. Noting that the argument of the logarithm should be strictly positive the domain of the inverse function is $D_{f^{-1}} =]0, \infty[$.

(b) The domain of f is $D_f =]0, \infty[$. To find the inverse function first note that $f(x) = \log \frac{3x}{2}$. So we have

$$y = \log \frac{3x}{2} \Rightarrow e^y = \frac{3x}{2} \Rightarrow x = \frac{2}{3}e^y$$

So the inverse function is given by $f^{-1}(x) = \frac{2}{3}e^x$ and $D_{f^{-1}} = \mathbb{R}$.

7. Compute

$$\lim_{x\to+\infty} \log(x).$$

Solution: $\lim_{x\to+\infty} \log(x)$ is the supremum of the range of the function $\log(x)$, that is, the supremum of the domain of its inverse, which is e^x. So

$$\lim_{x\to+\infty} \log(x) = \lim_{x\to+\infty} \log(x) = \sup \mathbb{R} = +\infty.$$

8. The limit

$$\lim_{n\to\infty} \frac{\cos(n)}{\log(n)}$$

is

(a) 0

(b) -1

(c) $+1$

(d) $+\infty$

Solution: (a) is correct. Note that

$$\frac{-1}{\log(n)} \leq \frac{\cos(n)}{\log(n)} \leq \frac{1}{\log(n)}$$

And $\log(n) \to \infty$ as $n \to \infty$. So by squeeze theorem, the limit of the sequence is 0.

9. Find maximum and minimum of the following functions

   (a) $f(x) = x$ in $[-\pi, \pi]$

   (b) $f(x) = \sin(x) + \cos(x)$ in $[0, \frac{2\pi}{3}]$

Solution:

(a) The function is strictly increasing, so the minimum is at $-\pi$ and the maximum at π

(b) The derivative is $f'(x) = \cos(x) - \sin(x)$ and it vanishes only in $x = \frac{\pi}{4}$. We compute

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad f(0) = 1, \quad f\left(\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}}{2}$$

So there is a maximum in $\frac{\pi}{4}$, and a minimum in $\frac{2}{3}\pi$.

10. Show that the derivative of the function

$$f(x) = \begin{cases} x^2 \cos \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

at $x = 0$ is zero and then find $f'(x)$. Is f' continuous?

Solution: We use the definition of derivative, So we write

$$x^2 \cos \frac{1}{x} = 0 + 0 \cdot (x - 0) + r(x)$$

And we must show that $\lim_{x \to 0} \frac{r(x)}{x-0} = 0$

$$\lim_{x \to 0} \frac{r(x)}{x} = \lim_{x \to 0} \frac{x^2 \cos \frac{1}{x}}{x} = \lim_{x \to 0} x \cos \frac{1}{x} = 0$$

And the final step holds because cos is a bounded function. Now since $f(x)$ has no singularities on $]-\infty, 0[$ and $]0, \infty[$ we may use the derivative formulas to compute $f'(x)$.

$$f'(x) = \begin{cases} 2x \cos \dfrac{1}{x} + \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We see that $\lim_{x \to 0^-} f'(x)$ and $\lim_{x \to 0^+} f'(x)$ do not exist (why?) even though $f'(0)$ does exist. It means that f is a differentiable function but its derivative is not continuous at $x = 0$.

11. State if the following are true or false.

    (a) The function $f : [0, +\infty[\to [1, +\infty[$ defined by $f(x) = x^3 - x + e^x$ is invertible.

    (b) Let $a, b \in \mathbb{R}$, $a < b$. Given a continuous function $f :]a, b[\to \mathbb{R}$ which is not monotone, there exists a point $x_0 \in (a, b)$ at which f admits a local minimum.

Solution:

(a) True. We compute $f'(x) = 3x^2 - 1 + e^x$. We observe that $f'(x) > 0$ for $x > 0$, so the function f is injective because it is strictly increasing. It has a minimum at $x = 0$, $f(0) = 1$. The supremum of the range is $\lim_{x \to +\infty} f(x) = +\infty$. So f is invertible.

(b) False. Take for example $f(x) = -x^2$, $a = -1$, $b = 1$.

Revision Exercises

12. Consider the bijective function $f :]1, \infty[\to] - \infty, -2[$ defined as $f(x) = \log(x) - 2x$. Then the derivative of the inverse function $f^{-1}(y)$ at $y = -2$ is

    (a) $(f^{-1})'(-2) = -1$

    (b) $(f^{-1})'(-2) = 1$

    (c) $(f^{-1})'(-2) = -\frac{2}{5}$

    (d) $(f^{-1})'(-2) = \frac{2}{5}$

Solution: (a) is correct. Using the formula for the derivative of the inverse function we have

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

First we compute $f'(x)$

$$f'(x) = \frac{1}{x} - 2$$

Then we should find x such that $f(x) = -2$, clearly $x = 1$. By substitution we get $(f^{-1})'(-2) = -1$.

13. For which of the following item you can prove, using the intermediate value theorem, that there exists a c in I such that $f(c) = k$.

    (a) $f(x) = \frac{x^2+8}{x}$, $k = 5$, $I = [1, 3]$

    (b) $f(x) = x^2 + x + 1$, $k = 2$, $I = [-2, 3]$

    (c) $f(x) = \frac{1}{2x-1}$, $k = 0$, $I = [0, 1]$

    (d) $f(x) = \frac{10}{x^2+1}$, $k = 8$, $I = [0, 1]$

Solution: (d) is correct. The function must be continuous on the interval $[a, b]$ and satisfy $f(a) < k < f(b)$ of $f(a) > k > f(b)$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sin\left(\frac{1-x^2}{1+x^2}\right)$$

then.

(a) $f'(x) = \cos\left(\frac{1-x^2}{1+x^2}\right)\frac{-4x}{1+x^2}$

(b) $f'(x) = \cos\left(\frac{1-x^2}{1+x^2}\right)\frac{-4x}{(1+x^2)^2}$

(c) $f'(x) = \sin\left(\frac{1-x^2}{1+x^2}\right)\frac{-4x}{(1+x^2)^2}$

(d) $f'(x) = \sin\left(\frac{1-x^2}{1+x^2}\right)\frac{-4x}{1+x^2}$

Solution: (b) is correct.

15. Let the function $f : \mathbb{R}\backslash\{1/2\} \to \mathbb{R}$ be defined by

$$f(x) = \frac{2x^3 + x^2 + 2x + 1}{2x - 1}$$

then

(a) f has at least one root in $[-1, 0]$

(b) f has at least one root in $[0, 1]$

(c) f has at least two roots in $[-1, 1]$

(d) f has no roots

Solution: (a) is correct. To use the intermediate value theorem, we need f to be continuous on a given interval, since the function is not defined at $x = 1/2$ so the intermediate value theorem cannot be applied on $[0, 1]$ and $[-1, 1]$. On the interval $[-1, 0]$, f is continuous and we see that $f(-1) > 0$, $f(0) < 0$. The intermediate value theorem suggests that f has a solution in $[-1, 0]$.

16. Let the function $f :]-1, 1[\backslash\{0\} \to \mathbb{R}$ be defined by $f(x) = \frac{x\log(1+x)}{\cos(x)-1}$. Let $g :]-1, 1[\to \mathbb{R}$ be an extension of f that is continuous at 0. Then

(a) g exist and $g(0) = -2$.

(b) g exist and $g(0) = 2$.

(c) g exist and $g(0) = 0$.

(d) f does not have a continuous extension at 0.

(*Hint: Note that* $\log(1) = 0$ *so* $\log(1 + x) = \log(1 + x) - \log(1)$. *Then use the definition of derivative.*)

Solution: (a) is correct. We need to find the limit f as $x \to 0$. We use polynomial expansion to get

$$\begin{aligned}
\lim_{x\to 0} \frac{x\log(1+x)}{\cos(x)-1} &= \lim_{x\to 0} \frac{x\log(1+x)}{\cos(x)-1} \cdot \frac{\cos(x)+1}{\cos(x)+1} \\
&= \lim_{x\to 0} \frac{\log(1+x)}{x} \cdot \frac{x^2}{-\sin^2(x)} \cdot (\cos(x)+1) \\
&= \left(\lim_{x\to 0} \frac{\log(1+x)-\log(1)}{x}\right) \cdot \lim_{x\to 0} \frac{x^2}{-\sin^2(x)} \cdot (\cos(x)+1) \\
&= \left(\lim_{x\to 0} \log(1+x)'\right) \cdot (-1) \cdot 2 = -2
\end{aligned}$$

17. Check if the following series are convergent

(a) $\sum_{n=0}^{+\infty} \frac{n}{(n+1)!}$

(b) $\sum_{n=0}^{+\infty} \frac{3n+1}{n^2(n+1)^2}$

Solution:

(a) We have $0 \leq \frac{n}{(n+1)!} \leq \frac{1}{n^2}$, so the series converges by the comparison test.

(b) We have $\frac{3n+1}{n^2(n+1)^2} = \frac{1}{n^3}\frac{3n+1}{n+2+\frac{1}{n}}$. The sequence $\frac{3n+1}{n+2+\frac{1}{n}}$ converges to 3, so for n big enough we have

$$0 \leq \frac{3n+1}{n^2(n+1)^2} \leq \frac{4}{n^3}$$

(We could have replaced 4 with any real number strictly bigger than 3). The series converge by the comparison criterion.

18. For what values of $t > 0$ the following series converges? If convergent, what is the limit?

$$\sum_{n=0}^{+\infty} \left(\frac{t}{t+1}\right)^{2n}$$

Solution: The series is a geometric series. Since $\left(\frac{t}{t+1}\right)^2$ is always strictly smaller than 1, the series is always convergent and the limit is

$$\frac{1}{1-\left(\frac{t}{t+1}\right)^2}.$$