Analysis I (English)
Fall Semester 2025–2026

Analysis 1 - Exercise Set 5

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Check if the sequence starting from $n = 1$ defined as $a_n = \frac{\sin(\frac{1}{n})}{n}$ is monotone, and if it converges or diverges.

Solution: Consider the unit circle. Since the sin function is defined as the y-coordinate of a point on the unit circle, we have that, if $0 \leq p \leq q \leq \frac{\pi}{2}$, we have $0 = \sin(0) \leq \sin(p) \leq \sin(q) \leq \sin(\frac{\pi}{2}) = 1$. Since $0 \leq \frac{1}{n} \leq 1 \leq \frac{\pi}{2}$ for every positive integer, we have that $0 \leq \sin(\frac{1}{n}) \leq \sin(1) \leq \sin(\frac{\pi}{2}) = 1$ for every positive integer. So, the numerator is bounded between 0 and 1. Then, as the denominator is always greater than or equal than 1, we get $a_n \leq \sin(\frac{1}{n})$. In particular, the sequence $\{a_n\}$ is bounded by 0 and 1.

For every positive integer n, we have $0 \leq \frac{1}{n+1} \leq \frac{1}{n} \leq \frac{\pi}{2}$. So, we have $0 \leq \sin(\frac{1}{n+1}) \leq \sin(\frac{1}{n}) \leq 1$. In particular, $\sin(\frac{1}{n})$ is decreasing. Dividing a positive decreasing sequence, such as $\sin(\frac{1}{n})$, by a positive increasing one, such as n, we obtain a decreasing sequence, so a_n is monotone decreasing (notice that here we need the sequence to be positive, as having negative signs may swap the order of inequalities). We conclude that a_n, being monotone and bounded, converges. Another way to prove the convergence is to observe that since $0 \leq \sin(\frac{1}{n}) \leq 1$ for all $n \geq 1$, then $0 \leq a_n \leq \frac{1}{n}$. So a_n converges to 0 by the Squeeze Theorem.

2. Check if the sequence

   (a) $a_n = \frac{n}{4n-1}$

   (b) $a_n = (-1)^n \frac{n^2+\pi}{n}$ starting from $n = 1$

   is monotone, and if it converges or diverges.

Solution:

(a) Since $a_{n+1} - a_n = -\frac{1}{(4n-1)(4n+3)} < 0$, the sequence a_n is monotone decreasing. It is bounded below by 0, hence it converges. Indeed, $a_n = \frac{1}{4-\frac{1}{n}}$, so for all $\varepsilon > 0$ we have $0 \leq \frac{1}{4-\frac{1}{n}} - \frac{1}{4} \leq \varepsilon$ for all $n \geq n_\varepsilon = \left[\frac{1}{16\varepsilon} + \frac{1}{4}\right] + 1$.

(b) It is not monotone as can be seen by the fact that it oscillates between being positive and negative. It also does not converge because the sequence is unbounded. This is seen by $|a_n| = \frac{n^2+\pi}{n} = n + \frac{\pi}{n}$.

3. Find the limit of the following sequences, if they exist:

   (a) $a_n = \dfrac{5n^2 - 3n + 2}{3n^2 + 7}$

(b) $a_n = (-1)^n \dfrac{\sqrt[4]{n}}{\sqrt[3]{n}}$

(c) $a_n = \frac{\sqrt{n}-n+n^2}{2n^2+n^{\frac{3}{2}}+n}$

(d) $a_n = \sin(\frac{1}{n}) + \frac{n-2}{n\sqrt{2}+77}$

Solution:

(a)

$$\lim_{n\to\infty} \frac{5n^2-3n+2}{3n^2+7} = \lim_{n\to\infty} \frac{5-3\frac{1}{n}+\frac{2}{n^2}}{3+\frac{7}{n^2}} = \frac{5-3\lim\limits_{n\to\infty}\frac{1}{n}+2\lim\limits_{n\to\infty}\frac{1}{n^2}}{3+7\cdot\lim\limits_{n\to\infty}\frac{1}{n^2}} = \frac{5}{3}$$

(b) We have

$$\lim_{n\to\infty} \left|(-1)^n \frac{\sqrt[4]{n}}{\sqrt[3]{n}}\right| = \lim_{n\to\infty} n^{\left(\frac{1}{4}-\frac{1}{3}\right)} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{12}}} = 0.$$

Since $-|a_n| \le a_n \le |a_n|$ for all n, by the Squeeze Theorem we have

$$\lim_{n\to\infty} (-1)^n \frac{\sqrt[4]{n}}{\sqrt[3]{n}} = 0 .$$

(c) Dividing numerator and denominator by n^2 we get

$$a_n = \frac{n^{-\frac{3}{2}} - n^{-1} + 1}{2 + n^{-\frac{1}{2}} + n^{-1}}$$

so, arguing as in part (a), the limit is $\frac{1}{2}$.

(d) As argued in problem 4, $\sin(\frac{1}{n})$ converges to 0. Furthermore, arguing as in part (a), $\frac{n-2}{n\sqrt{2}+77}$ converges to $\frac{1}{\sqrt{2}}$; both limits exist, so the sum converges to the sum of the limits, which is $\frac{1}{\sqrt{2}}$

4. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

   (a) If $\{a_n\}$ is bounded then $\{a_n\}$ is convergent.

   (b) If $\{a_n\}$ is bounded and $a_n \ge 0$, $\forall n \in \mathbb{N}$, then $\{a_n\}$ is convergent.

   (c) If $\{a_n\}$ is monotone and unbounded, then it is bounded from above.

   (d) If $\{a_n\}$ is monotone and unbounded, then it is bounded from below.

   (e) If $\{a_n\}$ is bounded and monotone then $\{a_n\}$ is convergent.

   (f) If $\{a_n\}$ is convergent, then there exists $\epsilon > 0$ such that $|a_n| \le \epsilon$ for all $n \in \mathbb{N}$.

   (g) Let $\{a_n\}$ be a sequence and let $\{b_n\}$ be the sequence defined as $b_n := |a_n|$. Then, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} b_n = 0$.

Solution:

(a) False, take the sequence $a_n = (-1)^n$.

(b) False, take the sequence $a_n = (-1)^n + 1$.

(c) False, take the sequence $a_n = n$.

(d) False, take the sequence $a_n = -n$.

(e) True. Suppose that a_n is monotone increasing. We prove that $\lim\limits_{n \to +\infty} a_n = \sup\limits_{n \in \mathbb{N}} a_n := a$.

Let $\varepsilon > 0$ be given. By the definition of supremum $\exists N \in \mathbb{N}$ s.t. $|a - a_N| = a - a_N < \varepsilon$. Let $n > N$, by monotonicity we have $|a - a_n| = a - a_n \leq a - a_N < \varepsilon$. Thus, $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a - a_n| < \varepsilon$ as required.

Now suppose that a_n is monotone decreasing. We prove that $\lim\limits_{n \to +\infty} a_n = \inf\limits_{n \in \mathbb{N}} a_n := a$.

Let $\varepsilon > 0$ be given. By the definition of infimum $\exists N \in \mathbb{N}$ s.t. $|a_N - a| = a_N - a < \varepsilon$. Let $n > N$, by monotonicity we have $|a_n - a| = a_n - a \leq a_N - a < \varepsilon$. Thus, $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a - a_n| < \varepsilon$ as required.

(f) True. Convergent sequences are bounded. If $a_n \to a$ then $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a_n - a| < \frac{1}{17} \Rightarrow |a_n| = |a_n - a + a| \leq |a_n - a| + |a| < \frac{1}{17} + |a|$. Thus, $\forall n > N \quad |a_n| < \frac{1}{17} + |a|$. It therefore follows that $|a_n| \leq \max\{|a_1|, \cdots, |a_{N-1}|, \frac{1}{17} + |a|\} \quad \forall n \in \mathbb{N}$. Hence, we may choose $\epsilon = \max\{|a_1|, \cdots, |a_{N-1}|, \frac{1}{17} + |a|\}$.

(g) True. 0 is the limit of $(a_n) \iff_{\text{by definition}} \forall \epsilon > 0$, $\exists n_\epsilon \in \mathbb{N}$ such that for $n \geq n_\epsilon$, $|a_n - 0| = |\frac{1}{n}| < \epsilon \iff_{\text{since } |\frac{1}{n}| = b_n} \forall \epsilon > 0$, $\exists n_\epsilon \in \mathbb{N}$ such that for $n \geq n_\epsilon$, $|b_n - 0| = |\frac{1}{n}| < \epsilon \iff_{\text{by definition}} \lim_{n \to \infty} b_n = 0$.

5. Let $p > q$ be natural numbers. Show that if $P(x) = \sum_{i=0}^{p} c_i x^i$ is a polynomial with real coefficients of degree p (that is, $c_p \neq 0$), and $Q(x) = \sum_{j=0}^{q} b_j x^j$ is a polynomial with real coefficients of degree q (that is, $b_q \neq 0$), then the sequence (a_n) defined as

$$a_n := \frac{P(n)}{Q(n)} \quad \text{is unbounded.}$$

Solution:

By the contrapositive of exercise 7(c) above, it suffices to show that for any positive real number C, there exists a natural number $n_C \in \mathbb{N}$ such that $|a_{n_C}| \geq C$. Let us assume by contradiction that $\{a_n\}$ is bounded. Then, there exists positive real number Csuch that for all $n \in \mathbb{N}$ then $|a_{n_C}| < C$. Then,

$$0 \leq \left(\frac{1}{n}\right)^{p-q} |a_n| \leq \left(\frac{1}{n}\right)^{p-q} C.$$

As $p > q$, then $\lim_{n \to \infty} \left(\frac{1}{n}\right)^{p-q} = 0$. Hence, by the Squeeze Theorem, the above 2 equations imply that $\lim_{n \to \infty} \left(\frac{1}{n}\right)^{p-q} |a_n| = 0$. On the other hand,

$$\left(\frac{1}{n}\right)^{p-q} |a_n| = |\frac{P(n)}{n^{p-q} Q(n)}| = |\frac{P(n)}{Q_1(n)}|,$$

where the polynomial $Q_1(x)$ is defined as $Q_1(x) = x^{p-q}Q(x)$. Then the degree of $Q_1(x)$ is exactly p and the coefficient of leading term of Q_1 (that is the coefficient of the monomial of degree p in Q_1) is $b_q \neq 0$. Hence, the result that we discussed in class, tell us that since P, Q_1 have the same degree then

$$\lim_{n\to\infty} |\frac{P(n)}{Q_1(n)}| = |\frac{a_p}{b_q}| \neq 0,$$

since $a_p \neq 0$. But this is contradiction with what we proved before.

6. Find the limit of the following sequences, if they exist:
   (a) $a_n = \sqrt{2n^2+3} - \sqrt{(2n+1)(n+4)}$
   (b) $a_n = \sqrt{n}(\sqrt{n^3+2n} - \sqrt{n^3+4})$

Solution:

(a)

$$\lim_{n\to\infty} \sqrt{2n^2+3} - \sqrt{(2n+1)(n+4)} = \lim_{n\to\infty} \frac{2n^2+3-(2n+1)(n+4)}{\sqrt{2n^2+3}+\sqrt{(2n+1)(n+4)}}$$
$$= \lim_{n\to\infty} \frac{-9n-1}{\sqrt{2}n\left(\sqrt{1+\frac{3}{2n^2}}+\sqrt{1+\frac{9}{2n}+\frac{2}{n^2}}\right)} = \frac{-9}{2\sqrt{2}}.$$

(b)

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n^3+2n} - \sqrt{n^3+4}) = \lim_{n\to\infty} \frac{\sqrt{n}(n^3+2n-n^3-4)}{\sqrt{n^3+2n}+\sqrt{n^3+4}}$$
$$= \lim_{n\to\infty} \frac{\sqrt{n}(2n-4)}{n\sqrt{n}\left(\sqrt{1+\frac{2}{n^2}}+\sqrt{1+\frac{4}{n^3}}\right)} = 1$$

7. Find the limit of the following sequences:
   (a) $a_n = \sin\left(\frac{1}{n}\right)$
   (b) $a_n = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$
   (c) $a_n = n \cdot \sin\left(\frac{2n+3}{n^3}\right)$

Hint: remember that for $0 < x < \pi/2$ *we have the inequalities:*

$$0 \le \sin(x) \le x \le \tan(x) \quad \Rightarrow \quad 1 \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)} \quad \Rightarrow \quad \cos(x) \le \frac{\sin(x)}{x} \le 1$$
$$\Rightarrow \quad \cos(x)^2 \le \left(\frac{\sin(x)}{x}\right)^2 \le 1 \quad \Rightarrow \quad 1-\sin(x)^2 \le \left(\frac{\sin(x)}{x}\right)^2 \le 1$$
$$\Rightarrow \quad 1-x^2 \le \left(\frac{\sin(x)}{x}\right)^2 \le 1 \quad \Rightarrow \quad \sqrt{1-x^2} \le \frac{\sin(x)}{x} \le 1\ .$$

Solution:

(a) We have

$$0 \leq \sin\left(\frac{1}{n}\right) \leq \frac{1}{n} \qquad \xRightarrow{\text{Squeeze Theorem}} \qquad \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0 \ .$$

(b) We have

$$\sqrt{1-\frac{1}{n^2}} \leq \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \leq 1 \qquad \xRightarrow{\text{Squeeze Theorem}} \qquad \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$$

(c) We have

$$\sqrt{1-\left(\frac{2n+3}{n^3}\right)^2} \leq \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}} \leq 1 \ .$$

Similarly to part (*b*), we first argue that

$$\lim_{n\to\infty} \sqrt{1-\left(\frac{2n+3}{n^3}\right)^2} = 1 \ .$$

According to the Squeeze Theorem

$$\lim_{n\to\infty} \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}} = 1 \ .$$

So

$$\begin{aligned}\lim_{n\to\infty}\left(n \sin\left(\frac{2n+3}{n^3}\right)\right) &= \lim_{n\to\infty}\left(\frac{2n+3}{n^2}\cdot\frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}}\right) \\ &= \left(\lim_{n\to\infty}\frac{2n+3}{n^2}\right)\cdot\left(\lim_{n\to\infty}\frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}}\right) = 0\cdot 1 = 0 \ .\end{aligned}$$

Note that we can split the limits because both sequences $\left(\frac{2n+3}{n^2}\right)$ and $\left(\frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}}\right)$ converge.

8. Show that the sequence given by

$$\begin{aligned} a_1 &= 2 \\ a_n &= \frac{1}{2}(a_{n-1}+6) \end{aligned}$$

is increasing and bounded above by 6. (*Hint: Use induction for both*)

Solution:

It is slightly easier to show first that the sequence is bounded and then that it is monotone, and we will indeed apply this strategy in the next exercises. Here, let's show first that it is monotone.

To show that the sequence is increasing we define the difference sequence as $d_n = a_{n+1} - a_n = 3 - a_n/2$. Showing $d_n \geq 0$ is equivalent to showing (a_n) is increasing. We show by induction: for $n = 1$, $d_1 = 2 > 0$. Now suppose that $d_n \geq 0$ meaning $a_n \leq 6$. For $n+1$ we have:

$$d_{n+1} = 3 - \frac{a_{n+1}}{2} = 3 - \frac{3 + a_n/2}{2} = \frac{3}{2} - \frac{a_n}{4} \geq \frac{3}{2} - \frac{6}{4} = 0$$

So (a_n) is increasing. To show that it is bounded above by 6 we use induction again. For $n = 1$ we have that $a_1 = 2 < 6$. Suppose that $a_n < 6$, for $n+1$ we have:

$$a_{n+1} = \frac{1}{2}(a_n + 6) \leq \frac{1}{2}(6 + 6) = 6.$$

9. Consider two sequences of real numbers a_n and b_n. Assume that $0 < a_n < 3$ and $-4 < b_n < 0$ for every n. Which of the following claims is true? (Only one choice is correct)

   (a) The sequence $\frac{1}{a_n}$ is bounded.
   (b) The sequence $a_n b_n$ is bounded below by -4.
   (c) The sequence $a_n + b_n$ has to be negative.
   (d) The sequence $a_n b_n$ is bounded below by -12.
   (e) The sequence $\frac{a_n}{b_n}$ is bounded.

Solution: The right answer is (d). Indeed, for every n, we have

$$-12 = -4 \cdot 3 < -4 \cdot a_n < b_n \cdot a_n,$$

where the first inequality comes from multiplying $a_n < 3$ both sides by -4, and the second inequality comes from multiplying $-4 < b_n$ both sides by the positive number a_n.

A counterexample to (a) is given by $a_n = \frac{1}{n}$.

A counterexample to (b) is given by the constant sequences $a_n = 2$ and $b_n = -3$.

A counterexample to (c) is given by the constant sequences $a_n = 2$ and $b_n = -1$.

A counterexample to (e) is given by $a_n = 1$ (constant) and $b_n = -\frac{1}{n}$.

10. Let z and w be two complex numbers. Which of the following statements is true? (Only one choice is correct)

   (a) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{w}$
   (b) $|z| = z \cdot \overline{z}$
   (c) $\mathrm{Im}(z + w) = \mathrm{Re}(i(z + w))$
   (d) $\mathrm{Re}(z + w) = \mathrm{Im}(i(z + w))$
   (e) $i\mathrm{Re}(z + w) = \mathrm{Im}(z + w)$

Solution: Let $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$.

The right answer is (d). Indeed, $z + w = (a + c) + i(b + d)$, so $\mathrm{Re}(z + w) = a + c$. Similarly, we have $i(z + w) = -(b + d) + i(a + c)$, and $\mathrm{Im}(i(z + w)) = a + c$.

Answer (a) is incorrect, since we need to conjugate the denominator as well to obtain a right formula. A counterexample to the formula given in (a) is given by $z = i$ and $w = i$.

Answer (b) is incorrect, since we need to take the square root of $z \cdot \overline{z}$ to get the absolute value of z.

The above computation for (d) shows that (c) is wrong, as the formula also needs a negative sign to hold true.

Finally, answer (e) is incorrect, since real and imaginary parts are real numbers.

11. Let $\{a_n\}$ be a sequence. Specify if the following statements are true or false. If you believe that the statement is true, you should give a proof, otherwise, provide a counterexample to the statement.

    (a) If
    $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$
    then $\{a_n\}$ converges.

    (b) If
    $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$
    then $\{a_n\}$ diverges.

Solution:

(a) False, take $a_n = n$.

(b) False, take for example $a_n = 1/n$. We have
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$
but $\lim_{n\to\infty} \frac{1}{n} = 0$.

12. Prove the following properties of the binomial coefficients:

    (a) Symmetry: $\binom{n}{k} = \binom{n}{n-k}$;

    (b) Binomial formula: Assuming the recurrence formula that you find in (c) below, prove that $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. [*Hint*: use induction on n. You may use the result from (c).]

    (c)* Recurrence: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Deduce from this that $\binom{n}{k} \in \mathbb{N}$; [*Hint*: use induction on n.]

Solution:

(a) $\binom{n}{k} = \frac{n!}{k!(n-k!)} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

(b) Prove by induction. The formula for $n = 0$ can be easily verified (remember $0! = 1$ and $\binom{0}{0} = 1$). Now suppose it holds for some natural number m. Hence,

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}$$

Now show that this implies that the formula holds for $m+1$.

$$\begin{aligned}
&(x+y)^{m+1}\\
=&(x+y)^m(x+y)\\
=&\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k}(x+y)\\
=&\sum_{k=0}^{m}\binom{m}{k}x^{k+1}y^{m-k}+\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k+1}\\
=&x^{m+1}+\sum_{k=0}^{m-1}\binom{m}{k}x^{k+1}y^{m-k}+y^{m+1}+\sum_{k=1}^{m}\binom{m}{k}x^k y^{m+1-k}\\
=&x^{m+1}+\sum_{k=1}^{m}\binom{m}{k-1}x^k y^{m-k+1}+\sum_{k=1}^{m}\binom{m}{k}x^k y^{m+1-k}+y^{m+1}\\
=&x^{m+1}+\sum_{k=1}^{m}\left(\binom{m}{k}+\binom{m}{k-1}\right)x^k y^{m+1-k}+y^{m+1}\\
=&x^{m+1}+\sum_{k=1}^{m}\binom{m+1}{k}x^k y^{m+1-k}+y^{m+1}\\
=&\sum_{k=0}^{m+1}\binom{m+1}{k}x^k y^{m+1-k}
\end{aligned}$$

as required.

(c) Follows from direct computation:

$$\begin{aligned}
&\binom{n-1}{k-1}+\binom{n-1}{k}\\
=&\frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!}\\
=&\frac{(n-1)!k!(n-k-1)!+(n-1)!(k-1)!(n-k)!}{k!(n-k)!(k-1)!(n-k-1)!}\\
=&\frac{1}{k!(n-k)!}\frac{(n-1)!k!(n-k-1)!+(n-1)!(k-1)!(n-k)!}{(k-1)!(n-k-1)!}\\
=&\frac{1}{k!(n-k)!}\frac{(n-1)!k+(n-1)!(n-k)}{1}\\
=&\frac{n(n-1)!}{k!(n-k)!}\\
=&\binom{n}{k}
\end{aligned}$$

as required. Now we can show that they are all contained in $\mathbb{N}$ by induction on n. One can verify that it holds for $n = 0$. Now suppose it holds for some $m - 1$. Then

$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$ must also be a natural number since it is a sum of two natural numbers.

13. If a sequence (x_n) converges, then its limit is unique.

Solution: Let (x_n) be a convergent sequence, and assume that both $x, y \in \mathbb{R}$ are limits of this sequence. We need to conclude that $x = y$.

By definition of convergence, for every $0 < \varepsilon \in \mathbb{R}$ there are $n_\varepsilon^x, n_\varepsilon^y \in \mathbb{N}$ such that for all $n \geq n_\varepsilon^x$ we have:

$$|x - x_n| \leq \varepsilon$$

and for all $n \geq n_\varepsilon^y$ we have

$$|y - x_n| \leq \varepsilon.$$

So, if we set $n_\varepsilon := \max\{n_\varepsilon^x, n_\varepsilon^y\}$, then both of the above inequalities hold for all integers $n \geq n_\varepsilon$. In particular, for such n, we have

$$|y - x| = \underbrace{|y - x_n + x_n - x| \leq |y - x_n| + |x_n - x|}_{\text{triangle inequality}} \leq \varepsilon + \varepsilon = 2\varepsilon$$

Since, this holds for all $0 < \varepsilon \in \mathbb{R}$, we obtain that $y = x$.

14. Assume that $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$. Prove the following fact: for any $l \in \mathbb{N}$, $\lim\limits_{n\to\infty} x_{n+l}$ exists and $\lim\limits_{n\to\infty} x_{n+l} = x$.

Solution: Let us fix $l \in \mathbb{N}$. Let us define the sequence (y_n), $y_n := x_{n+l}$. Then, we need to prove that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$. If $l = 0$, then there is nothing to prove, since $x_n = y_n$, $\forall n \in \mathbb{N}$. Hence, we can assume that $l > 0$. By definition of limit, for any $\varepsilon > 0$ there exists $n'_\varepsilon > 0$ such that

$$\forall n \geq n'_\varepsilon, \text{ then } |x_n - x| < \varepsilon.$$

As $x_n = y_{n-l}$, then

$$\forall n \geq n'_\varepsilon, \text{ then } |y_{n-l} - x| < \varepsilon.$$

Hence, we can rewrite the above by saying that $\forall n \geq n'_\varepsilon - l$, then $|y_n - x| < \varepsilon$. Hence, taking $n_\varepsilon := n'_\varepsilon - l$, we see that x satisfies the definition of limit for y_n.