Analysis 1 - Exercise Set 2

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Let $p \in \mathbb{N}$ be a prime number. Prove that $\sqrt{p}$ is not rational.

   (b) Show that $\sqrt{7+\sqrt{17}}$ is irrational. (*Hint: Use part (a) to prove that $\sqrt{17}$ is irrational. Now assume that $\sqrt{7+\sqrt{17}}$ is rational and show that it contradicts the fact that $\sqrt{17}$ is irrational.*)

   (c) Show that $\sqrt{2}+\sqrt[3]{3}$ is irrational. (*Hint: Let $r=\sqrt{2}+\sqrt[3]{3}$ and assume it is rational. Compute $(r-\sqrt{2})^3$ and use the result that you obtained plus the assumption on the rationality of r to find a contradiction.*)

Solution:

(a) Let $\sqrt{p}=\frac{a}{b}$ such that $\gcd(a,b)=1$. We square both sides to get

$$p=\frac{a^2}{b^2}$$

Now we multiply both sides with b^2 to get

$$a^2=pb^2$$

Clearly p is a factor of right hand side so p must be a factor of left hand side too. But since p is a factor of a^2, a is an integer and p is a prime number, we deduce that p is also a factor of a. Hence we can find an integer c such that $a=pc$. By replacing this in the above equation we get

$$a^2=p^2c^2=pb^2 \implies b^2=pc^2$$

Using an argument similar to before we can deduct that p is also a factor of b. This is a contradiction since we assumed that $\gcd(a,b)=1$ but p divides both a and b.

(b) Let $r=\sqrt{7+\sqrt{17}}$. Since 17 is a prime number, $\sqrt{17}$ is irrational by part (a). Now we can rewrite r as

$$r^2-7=\sqrt{17}$$

If r is rational then so is r^2-7 and this is a contradiction since $\sqrt{17}$ is irrational.

(c) Let $r=\sqrt{2}+\sqrt[3]{3}$. We have

$$(r-\sqrt{2})^3=3$$

and then

$$0=r^3-3r^2\sqrt{2}+6r-2\sqrt{2}-3=r^3+6r-3-\sqrt{2}(3r^2+2).$$

So

$$\sqrt{2}=\frac{r^3+6r-3}{3r^2+2}.$$

If r is a rational number then the right hand side becomes rational. This contradicts the fact that $\sqrt{2}$ is irrational.

2. Let S be a subset of $\mathbb{R}$. Let a be a lower bound (respectively an upper bound) for S. Show that any real number b such that $b < a$ (respectively $b > a$) then b is also a lower bound (resp. an upper bound) for S.

> **Solution:**
> Lower bound case: Let a be a lower bound for $S \Leftrightarrow \forall x \in S \quad b < a \leq x \Rightarrow \forall x \in S \quad b < x \Rightarrow b$ is a lower bound for S.
> Upper bound case: Let a be an upper bound for $S \Leftrightarrow \forall x \in S \quad b > a \geq x \Rightarrow \forall x \in S \quad b > x \Rightarrow b$ is an upper bound for S.

3. Let A be a bounded interval in $\mathbb{R}$, i.e., A is a subset of $\mathbb{R}$ of either one of the following forms: $[a, b]$, or $]a, b[$, or $[a, b[$, or $]a, b]$, with $a, b \in \mathbb{R}$ and $a < b$. State if the following statements are true or false. If you true, explain why. If false, find an example of an interval that contradicts that statement.

   (a) $\sup(A) \in A$ and $\inf(A) \in A$.
   (b) If $\sup(A) \in A$ and $\inf(A) \in A$ then A is closed.
   (c) If A is closed then $\sup(A) \in A$ and $\inf(A) \in A$.
   (d) If $\sup(A) \notin A$ and $\inf(A) \notin A$ then A is open.
   (e) If A is open then $\sup(A) \notin A$ and $\inf(A) \notin A$.

> **Solution:**
>
> (a) False. For example, take the interval $A = [1, 2[$.
>
> (b) True. Verification: Assume that it is false. Then A is not closed, for example $A = [a, b[= \{x \in \mathbb{R} : a \leq x < b\}$. We observe that b is an upper bound and that if $c < b$ is another upper bound, then $c > a$ and we know from the lecture that there exists a real number d such that $c < d < b$. Then $d \in A$, so c cannot be an upper bound. This shows that $\sup(A) = b$. But $b \notin A$. Hence, we reached a contradiction. A similar argument for $\inf(A) = a$ works if $A =]a, b[$ or $A =]a, b]$.
>
> (c) True. If A is closed there exist real numbers $a < b$ such that $A = [a, b]$. Then, $a = \min(A) = \sup(A)$ and $b = \max(A) = \sup(A)$.
>
> (d) True. Verification: Assume that it is false. Then A is not open, for example $A = [a, b[= \{x \in \mathbb{R} : a \leq x < b\}$. An argument as above shows that $\inf(A) = a$. But $a \in A$. Hence, we have a contradiction. A similar argument for $\sup(A) = b$ works if $A = [a, b]$ or $A =]a, b]$.
>
> (e) True. If A is open there exist real numbers $a < b$ such that $A =]a, b[$. An argument as in the previous case shows that $\sup(A) = b$ and $\inf(A) = a$. Hence, they do not belong to A.

4. Let A be a bounded interval in $\mathbb{R}$, i.e., A is a subset of $\mathbb{R}$ of either one of the following forms: $[a, b]$, or $]a, b[$, or $[a, b[$, or $]a, b]$, with $a, b \in \mathbb{R}$ and $a < b$. Show that $\inf A = a$, $\sup A = b$. When is the infimum (resp. maximum) of A a minimum (resp. a maximum)?

Solution: Let us show that $\inf A = a$. We first show that a is a lower bound for A. This follows since $\forall d \in A$,

$$\begin{cases} d \geq a, & \text{if } A = [a, b] \text{ or } A = [a, b[\\ d > a, & \text{if } A =]a, b] \text{ or } A =]a, b[. \end{cases}$$

We need to show that a is the largest lower bound for A, that is, we need to show that if c is a real number such that $c > a$, then c is not a lower bound for A. To show this, it suffices to show that there exists an element l of A such that $l < c$. Since $c > a$, then $a < a + \frac{c-a}{2} < c$. If $a + \frac{c-a}{2} \in A$, it suffices to take $l := a + \frac{c-a}{2}$. If $a + \frac{c-a}{2} \notin A$, then $c > b$, and it suffices to take $l := b$.

The proof that $\sup A = b$ is similar. We first show that b is an upper bound for A. This follows since $\forall e \in A$,

$$\begin{cases} e \leq b, & \text{if } A = [a, b] \text{ or } A =]a, b] \\ e < b, & \text{if } A =]a, b[\text{ or } A = [a, b[. \end{cases}$$

We need to show that b is the smaller upper bound for A, that is, we need to show that if f is a real number such that $f < b$, then f is not a lower bound for A. To show this, it suffices to show that there exists an element m of A such that $m > f$. Since $f < b$, then $f < b + \frac{f-b}{2} < b$. If $b + \frac{f-b}{2} \in A$, it suffices to take $m := b + \frac{f-b}{2}$. If $b + \frac{f-b}{2} \notin A$, then also $f < a$, and it suffices to take $m := a$.

In view of the above, then A has a minimum (resp. maximum) if and only if $a \in A$ (resp. $b \in A$). Hence A has has a minimum (resp. maximum) if and only if $A = [a, b]$ or $A = [a, b[$ (resp. $A =]a, b]$ or $A = [a, b]$).

5. Let S be a subset of $\mathbb{R}$. Show that if $\sup(S)$, $\inf(S)$, $\max(S)$, $\min(S)$ exist, then they are unique.

Solution: Suppose that the supremum for a non-empty set $S \subset \mathbb{R}$ is not unique. Then there are at least two 'numbers' $a < b$ (we allow $b = +\infty$) s.t. they are the supremum of S, so they are both a smallest upper bound for S. But b can obviously not be the supremum since it cannot be the smallest upper bound, since a is an upper bound smaller than b. Hence, our assumption that there are more than one suprema of S must be false. Apply a similar argument for the infimum.

The results for maximum and minimum are due to the fact that they are special cases of supremum and infimum respectively.

6. Let $S \subseteq \mathbb{R}$ be the subset of the real numbers defined as $S := \{x \in \mathbb{R} \mid x \in \mathbb{Q} \text{ and } x^3 \geq 5\}$.
   (a) Show that S is not empty (i.e., exhibit an element of S).
   (b) Show that $\sqrt[3]{5}$ is a lower bound for S.
   (c) Show that $\inf(S) = \sqrt[3]{5}$. *(Hint: you should use the denseness of $\mathbb{Q}$ in this step).*

   Hint: you can use the fact that every real number has a unique real cubic root, and that $a^3 \leq b^3$ if and only if $a \leq b$.

Solution:

(a) Consider 10. As it is a natural number, it is a rational number. Furthermore, $10^3 = 1000 > 5$. Thus, $10 \in S$.

(b) To show that $\sqrt[3]{5}$ is a lower bound, we have to show the following:

$$\forall x \in S, x \geq \sqrt[3]{5}.$$

Fix $x \in S$. Then, by definition of S, we have $x^3 \geq 5$. By the hint, this fact is equivalent to $x \geq \sqrt[3]{5}$. This concludes the proof of (a).

(c) We have showed that $\sqrt[3]{5}$ is a lower bound. To show it is the infimum, we have to show the following:

$$\forall \epsilon > 0, \exists x \in S, \sqrt[3]{5} \leq x \leq \sqrt[3]{5} + \epsilon.$$

Fix $\epsilon > 0$. Then, we have $\sqrt[3]{5} < \sqrt[3]{5} + \epsilon$. By denseness of $\mathbb{Q}$, there exists $x \in \mathbb{Q}$ such that $\sqrt[3]{5} < x < \sqrt[3]{5} + \epsilon$. By the hint, the first inequality, i.e., $\sqrt[3]{5} < x$, is equivalent to $x^3 > 5$. Since $x \in \mathbb{Q}$, this shows that $x \in S$. Thus, we found the sought element of S in the interval $[\sqrt[3]{5}, \sqrt[3]{5} + \epsilon]$. This concludes the proof.

7. For each of the following sets, check if they are bounded or unbounded. When the set is bounded from above or below, give a few examples of lower and upper bounds, then compute the supremum and infimum and check if maximum and minimum exist.

(a) $A = \{x \in \mathbb{R} \mid x^2 \leq 2\}$.

(b) $B = \{x \in \mathbb{R} \mid x \in \mathbb{Q} \text{ and } x^2 \leq 2\}$.

(c) $C = \{(-1)^n + \frac{1}{n+1} | n \in \mathbb{N}\}$.

Solution:

(a) This set is bounded. In fact, -2 (resp. 2) is a lower bound (resp. upper bound). We prove the case of -2, the other is completely analogous. To show that -2 is a lower bound, we need to show that any $a \in A$ satisfies $-2 \leq a$. If $a \geq 0$, there is nothing to prove. If $a < 0$, then it suffices to observe that for any $x < y < 0$, then $0 < y^2 < x^2$ and this just follows from the definition of square of a real number. Hence, $a^2 \leq 2 < (-2)^2$ implies that $-2 < a$. As that holds for any $a \in A$, $a < 0$, we are done.

We claim that the supremum is $\sqrt{2}$ and infimum $-\sqrt{2}$. As $\pm\sqrt{2} \in A$, then they are also maximum and minimum, respectively. Let us prove that $\sqrt{2}$ is the supremum, the other case is analogous. As for positive numbers

$$0 < s < t \implies 0 < s^2 < t^2, \tag{1}$$

then, since $(\sqrt{2})^2 = 2$, it follows that $z \leq \sqrt{2}$ for any $z \in A$. Hence, $\sqrt{2}$ is an upper bound for A. Let us assume that $\sqrt{2}$ is not the smallest of the upper bounds, that is $\inf A < \sqrt{2}$. Let $l' := \sup A$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $c \in \mathbb{Q}$ such that

$$l' < c < \sqrt{2}.$$

Then $c^2 < 2$, hence $c \in A$ and $l' < c$ which gives a contradiction, since we assumed that l' is the supremum of A.

(b) This set is bounded, as it is a subset of the set in part (a). By what showed in part (a), $\sqrt{2}$ is an upper bound and $-\sqrt{2}$ is a lower bound. We claim that $\sqrt{2} = \sup(B)$ (the argument for $-\sqrt{2} = \inf(B)$ is analogous). By denseness of $\mathbb{Q}$, for every $\epsilon > 0$, we can find a rational number r such that $\sqrt{2} - \epsilon < r_n < \sqrt{2}$. Then, $r_n \in B$, as it is positive, rational, and $r_n^2 < 4$. Thus, $\sqrt{2}$ satisfies the characterization of supremum given in class. Since $\sup(B) = \sqrt{2}$, $B \subset \mathbb{Q}$ and $\sqrt{2} \notin \mathbb{Q}$, it follows that B has no maximum. Similarly, as $\inf(B) = -\sqrt{2}$, B has no minimum.

(c) The set is bounded because -1 is a lower bound and 2 is an upper bound, $\sup(C) = 2$, $\inf(C) = -1$, the maximum is 2 but the minimum does not exist. To verify that 2 is the maximum and the supremum we observe that it is an upper bound and that $(-1)^n + \frac{1}{n+1} = 2$ if $n = 0$.

To verify that $\inf(C) = -1$, we observe that if $a > -1$, then $a > (-1)^{2m+1} + \frac{1}{2m+2} = -1 + \frac{1}{2m+2}$ is satisfied for $2m+1 > \frac{1}{a+1} - 1$. So, no number $a > -1$ is a lower bound for C. Thus, as -1 is a lower bound, it is the infimum.

To verify that the minimum does not exist, we observe that $(-1)^n + \frac{1}{n+1} > -1$ for all $n \in \mathbb{N}$ (treat the cases $n = 0$, $n \geq 1$ odd, $n \geq 1$ even separately).

8. Let a be a real number. Assume that $a \geq 0$. Prove that $a = 0$ if and only if for any $\varepsilon > 0, a \leq \epsilon$.

Solution:
($\Rightarrow$) If $a = 0$ then it is smaller than any positive ε.
($\Leftarrow$) Suppose that $\forall \varepsilon > 0 \quad a \leq \varepsilon$ and $a \in \mathbb{R}_+$. Assume $a > 0$. Then $\exists \varepsilon > 0$ (take $\varepsilon = \frac{a}{2}$) s.t. $a > \varepsilon = \frac{a}{2} > 0$. This leads to a contradiction, hence our assumption that $a > 0$ must be false. Since a is non-negative we must have $a = 0$.

9. Let L and L' be two real numbers. Prove that the following are equivalent:

(a) $L = L'$;

(b) for every $\epsilon > 0$, $|L - L'| \leq \epsilon$.

Solution:
($\Rightarrow$) If $L = L'$, then $|L - L'| = 0$. Thus, given any positive number ϵ, $|L - L'| = 0 \leq \epsilon$.
($\Leftarrow$) Suppose that for every $\epsilon > 0$, $|L - L'| \leq \epsilon$. Assume by contradiction that $L \neq L'$. Then, by definition of absolute value, $|L - L'| > 0$, as the absolute value is always non-negative and it is 0 only if the input is 0. Now, choose $\epsilon = \frac{|L-L'|}{2}$. Then, by our assumption, we have $|L - L'| \leq \epsilon = \frac{|L-L'|}{2}$. This is absurd, as $|L - L'| \leq \frac{|L-L'|}{2}$ is false, as $|L - L'| > 0$.

10. Let $S \subseteq \mathbb{R}$ be a non-empty subset. Assume that S is bounded from above and that $\sup(S) \notin S$. Show that the following fact holds: for every $\epsilon > 0$, $S \cap]\sup(S) - \epsilon, \sup(S)[$ is infinite (i.e., there are infinitely many elements of S in $]\sup(S) - \epsilon, \sup(S)[$).
Hint: In this problem, you can freely use that a finite non-empty set has both maximum and minimum.

Solution: We argue by contradiction. Let S be a set as in the statement, and assume that the claim does not hold. Then, there exists $\epsilon > 0$ such that $S \cap]\sup(S) - \epsilon, \sup(S)[$ is finite. Fix this $\epsilon > 0$. By the characterization of the supremum given in homework 2, we know $S \cap]\sup(S) - \epsilon, \sup(S)[$ is not empty. So, $S \cap]\sup(S) - \epsilon, \sup(S)[$ is non-empty and finite, and we can consider its maximum s_0. Now, define $\delta = \frac{\sup(S) - s_0}{2}$, i.e., half of the distance between s_0 and $\sup(S)$. Notice that this is a positive number, as $s_0 < \sup(S)$. Notice that, as $s_0 \in]\sup(S) - \epsilon, \sup(S)[$, we have $\sup(S) - s_0 < \epsilon$. Thus, we have $0 < \delta < \epsilon$.

Now, we use the characterization of the supremum given in class. Thus, there is $s' \in S$ such that $\sup(S) - \delta \leq s' \leq \sup(S)$. Since we know $\sup(S) \notin S$, we also know that $s' \neq \sup(S)$, thus $s' < \sup(S)$. Fix such s'. Then, we have

$$\sup(S) - \epsilon < s_0 = \sup(S) - 2\delta < \sup(S) - \delta \leq s' < \sup(S).$$

In particular, $s' \in (\sup(S) - \epsilon, \sup(S))$ and $s' > s_0$. This contradicts the fact that s_0 is the maximum of $S \cap]\sup(S) - \epsilon, \sup(S)[$. This provides the sought contradiction, and the claim follows.

Alternative way to argue (only sketch): alternatively, once we produce the number s_0 as above, one can show that $s_0 = \max(S)$, and this would give a contradiction as well.

11. Let S be a non-empty and bounded subset of $\mathbb{R}$. We define

$$S' := \{x \in \mathbb{R} | -x \in S\}.$$

Show that

(a) If M is an upper bound of S, then $-M$ is a lower bound of S'.

(b) If m is a lower bound of S, then $-m$ is an upper bound of S'.

(c) $\sup(S) = -\inf(S')$.

(d) $\inf(S) = -\sup(S')$.

Solution:

(a) If M is an upper bound of S, then by definition $x \leq M$ for all $x \in S$. This means that, $-x \geq -M$ for all $x \in S$, and so $y \geq -M$ for all $y \in S'$. This shows that $-M$ is a lower bound of S';

(b) If m is a lower bound of S, then by definition $x \geq m$ for all $x \in S$. Then $-x \leq -m$ for all $x \in S$ and so $y \leq -m$ for all $y \in S'$. This shows that $-m$ is an upper bound of S';

(c) Let $b = \sup(S)$. Then $[b, +\infty[$ is the set of all the upper bounds for S. According to part (a) the set $] - \infty, -b[$ consists of all the lower bounds for S'. By definition, $\inf(S')$ is the greatest lower bound, so $\inf(S') = -b = -\sup(S)$;

(d) Let $b = \inf(S)$. Then $] - \infty, b]$ is the set of all the lower bounds for S. According to part (b) the set $[-b, \infty[$ consists of all the upper bounds for S'. By definition, $\sup(S')$ is the smallest lower bound, so $\sup(S') = -b = -\inf(S)$.

12. Let S be the subset of $\mathbb{R}$ defined as

$$S := \bigcap_{n=1}^{\infty} [0, 1 + \frac{1}{n}]$$

Compute $m := \sup S$. Is m the maximum of S? (*Hint*: $x \in S \iff \forall n \in \mathbb{N}, x \in [0, 1 + \frac{1}{n}]$)

Solution: First, observe that $0 \le 1 \le 1 + \frac{1}{n}$ for every $n \in \mathbb{N}$. Thus, $1 \in S$. Furthermore, for every $n \in \mathbb{N}$, $1+\frac{1}{n}$ is an upper bound: indeed, $S \subseteq [0, 1+\frac{1}{n}]$ and $1+\frac{1}{n} = \max([0, 1+\frac{1}{n}])$. Now, we claim that $1 = \sup(S)$. Assume by contradiction this is not the case. Then, as $1 \in S$, $m > 1$. Then, if we choose a natural number $n > \frac{1}{m-1}$, we have $1 + \frac{1}{n} < m$. Since $1 + \frac{1}{n}$ is an upper bound, m is not the least upper bound of S, contradicting the fact that $m = \sup(S)$. Thus, $m = 1$. Since $1 \in S$, we have $m = \max(S)$.

13. (Multiple choice) The subset S of $\mathbb{R}^2$ defined as[1]

$$S := \{(x, y) \in \mathbb{R}^2 \mid x = -y, -y = x - 1\}$$

is:

(a) A point.
(b) A line.
(c) A circle.
(d) Empty.

Solution:
(d) is correct. We must have $x = -y$ and $-y = x - 1$ which has no solutions since the system of equations

$$\begin{cases} x = -y \\ -y = x - 1 \end{cases}$$

implies that $-1 = 0$, clearly impossible. We conclude that no point in $\mathbb{R}^2$ can satisfy both the equations defining S and so, $S =$.

14. (Multiple choice) The subset S of $\mathbb{R}^2$ defined as

$$S := \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + (y+3)^2} = 3\sqrt{x^2 + y^2}\}$$

is:

(a) A point.
(b) A line.
(c) A circle.
(d) Empty.

Solution:
(c) is correct. We square both terms and write and we obtain

$$x^2 + (y+3)^2 = 9(x^2 + y^2) \iff x^2 + y^2 - \frac{3}{4}y = \frac{9}{8} \iff x^2 + \left(y - \frac{3}{8}\right)^2 = \left(\frac{9}{8}\right)^2.$$

Then the solution are all the points of the circle of radius 9/8 centered at (0, 3/8).

[1] In this exercise (x, y) does not denote an open interval between x and y, but it instead denotes the point of coordinates x and y in $\mathbb{R}^2$