Solutions to Example open questions

Problem 1 Show by induction that for all $n \geq 1$

$$\sum_{k=1}^{n} \frac{2^k (k-1)}{(k+1)!} = 2 - \frac{2^{n+1}}{(n+1)!}$$

<u>Solution:</u> We will prove the equality by induction.

Base case $n=1$:

Left hand side: $\frac{2\cdot(1-1)}{(1+1)!} = 0$

Right hand side: $2 - \frac{2^{1+1}}{(1+1)!} = 2-2=0$

So we get an equality.

Induction hypothesis:

Assume $\sum_{k=1} \frac{2^k (k-1)}{(k+1)!} = 2 - \frac{2^{n+1}}{(n+1)!}$ for any $n \leq L-1$

Induction step: We need to show that

$$\sum_{k=1}^{L} \frac{2^k (k-1)}{(k+1)!} = 2 - \frac{2^{L+1}}{(L+1)!}$$

Since we assume that the equality is true for $n = L-1$ we get:

$$\sum_{k=1}^{L} \frac{2^k (k-1)}{(k+1)!} = \sum_{k=1}^{L-1} \frac{2^k (k-1)}{(k+1)!} + \frac{2^L (L-1)}{(L+1)!} =$$

$$= \left(2 - \frac{2^L}{L!}\right) + \frac{2^L (L-1)}{(L+1)!} =$$

$$= 2 + \frac{2^L (L-1) - 2^L (L+1)}{(L+1)!} =$$

$$= 2 + \frac{2^L \left(L-1 - (L+1)\right)}{(L+1)!} =$$

$$= 2 - \frac{2^{L+1}}{(L+1)!}$$

which finishes the proof

$\blacksquare$

Problem 2: Let I be open interval and $f: I \to \mathbb{R}$ continuous at $x_0 \in I$.

Show that if $f(x_0) > 0$ then $f(x) > 0$ on some open interval containing x_0.

1st Solution:

By definition of continuity we have

$$\lim_{x \to x_0} f(x) = f(x_0) > 0$$

which means that for any $\varepsilon > 0$

$\exists \delta > 0$ s.t. $\forall x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$

$$|f(x) - f(x_0)| < \varepsilon .$$

Lets take $\varepsilon = \frac{f(x_0)}{2}$ then

$\exists\, \delta$ s.t. $\forall\ x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$

$$|f(x) - f(x_0)| < \frac{f(x_0)}{2}.$$

In particular $f(x) > 0$ for any

$x \in (x_0 - \delta, x_0 + \delta)$ which finishes the proof ∎

2nd Solution:

Assume counter-positive:

$\forall n \in \mathbb{N}_{>0}$ $\exists x_n$ with $|x_n - x_0| < \frac{1}{n}$ and $f(x_n) \leq 0$.

But then we have sequence of elements (x_n) converging to x_0 s.t. $f(x_n) \leq 0$.

And therefore $\lim\limits_{n \to \infty} f(x_n) \leq 0$.

On the other hand by continuity of $f(x)$ at x_0 we get $\lim_{x \to x_0} f(x) = f(x_0)$ which means that $\forall (a_n)$ s.t. $\lim_{n\to\infty} (a_n) = x_0$ and $a_n \neq x_0$ we have $\lim_{n\to\infty} [f(a_n)] = f(x_0) > 0$

So we get a contradiction. $\blacksquare$