Analysis I (English)
Fall Semester 2025–2026

Analysis 1 - Exercise Set 8

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Using the definition, state if the following functions are injective, surjective or bijective. If the function is bijective, find the inverse function.
   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^5$
   (b) $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$

Solution:

(a) Injective: we must show that if for some $x_1 \in x_2$ in the domain of f, $f(x_1) = f(x_2)$ then $x_1 = x_2$. We have

$$f(x_1) = f(x_2) \Rightarrow x_1^5 = x_2^5 \Rightarrow \sqrt[5]{x_1^5} = \sqrt[5]{x_2^5} \Rightarrow x_1 = x_2.$$

So f is injective.

Surjective: we must show that for any $y \in \mathbb{R}$ there exist x in the domain of f such that $f(x) = y$. For any given y it is enough to take $x = \sqrt[5]{y}$, then $f(x) = y$. So f is surjective.

Since f is both injective and surjective then it is bijective. The inverse function is given by $f^{-1}(x) = \sqrt[5]{x}$.

(b) Injective: We have

$$f(x_1) = f(x_2) \Rightarrow \sqrt{x_1} = \sqrt{x_2} \Rightarrow (\sqrt{x_1})^2 = (\sqrt{x_2})^2 \Rightarrow |x_1| = |x_2| \Rightarrow x_1 = x_2.$$

The last step is true because we know that $x_1, x_2 \in [0, \infty)$. So f is injective.

Surjective: This function is not surjective since there is no $x \in [0, \infty)$ that is mapped to negative numbers. Since the function is not surjective then it is not invertible.

2. For the two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ below, find $g \circ f$ and $f \circ g$.

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}, \qquad g(x) = \begin{cases} 2x-3 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$$

3. State if the following are true or false.
   (a) The function $f = \sqrt{1 - \cos x}$ is even.
   (b) There is no function which is both even and odd.
   (c) Let f be an odd function. If f is bijective, then f^{-1} is also odd.

Solution:

(a) True. We have

$$f(-x) = \sqrt{1 - \cos(-x)} = \sqrt{1 - \cos(x)} = f(x)$$

(b) False. If a function is both even an odd then we have:

$$f(-x) = f(x) = -f(x) \Longrightarrow f(x) = -f(x) \Longrightarrow f(x) = 0$$

So $f(x) = 0$ is a function that is both even and odd.

(c) True. Note that $f \circ f^{-1} = id$, which means that $f \circ f^{-1}(y) = id(y) = y$. We have

$$-y = -y \Rightarrow id(-y) = -id(y) \Rightarrow f \circ f^{-1}(-y) = -f \circ f^{-1}(y)$$

But since f is an odd function we have that $f(-x) = -f(x)$. So we can write

$$\Rightarrow f(f^{-1}(-y)) = f(-f^{-1}(y))$$

Now since f is injective we know that if $f(x_1) = f(x_2)$ then $x_1 =_2$, meaning

$$\Rightarrow f^{-1}(-y) = -f^{-1}(y)$$

This shows that f^{-1} is an odd function.

4. Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, Determine the monotonicity (increasing or decreasing) of the composition $g \circ f : \mathbb{R} \to \mathbb{R}$ in the following cases:
   (a) if f and g are both increasing.
   (b) if f and g are both decreasing.
   (c) if f is increasing and g is decreasing. What can we say about $f \circ g$?

Solution:

(a) If f and g are both increasing, we have

$$x_1 \leq x_2 \Longrightarrow f(x_1) \leq f(x_2) \Longrightarrow g(f(x_1)) \leq g(f(x_2))$$

In the first step we used monotonicity of f and in the second step we used monotonicity of g. So $g \circ f$ is an increasing function.

(b) If f and g are both decreasing, we have

$$x_1 \leq x_2 \Longrightarrow f(x_1) \geq f(x_2) \Longrightarrow g(f(x_1)) \leq g(f(x_2))$$

In the first step we used monotonicity of f and in the second step we used monotonicity of g. So $g \circ f$ is an increasing function.

(c) If f is increasing and g is decreasing, we have

$$x_1 \leq x_2 \Longrightarrow f(x_1) \leq f(x_2) \Longrightarrow g(f(x_1)) \geq g(f(x_2))$$

so $g \circ f$ is decreasing.

For $f \circ g$ we have

$$x_1 \leq x_2 \Longrightarrow g(x_1) \geq g(x_2) \Longrightarrow f(g(x_1)) \geq f(g(x_2))$$

This shows that $f \circ g$ is again decreasing. Hence, the composition of functions with opposite monotonicity is always decreasing.

5. Using the definition, state if the following functions are injective, surjective or bijective. If the function is bijective, find the inverse function.
   (a) $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin x$
   (b) $f : [0, \pi] \to [-1, 1]$, $f(x) = \cos x$

Solution:

(a) Injective: This function is not injective, because we can find $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. For example $\sin(0) = \sin(\pi) = 0$.

Surjective: For any given y it is enough to take $x = \arcsin(y)$ then $f(x) = y$. So f is surjective.

(b) Injective: We have

$$f(x_1) = f(x_2) \Rightarrow \cos(x_1) = \cos(x_2) \Rightarrow \arccos(\cos(x_1)) = \arccos(\cos(x_2)) \Rightarrow x_1 = x_2.$$

So f is injective.

Surjective: For any given $y \in [-1, 1]$ it is enough to take $x = \arccos(y)$, then $f(x) = y$. So f is surjective.

Since f is both injective and surjective then it is bijective. The inverse function is given by $f^{-1}(x) = \arccos(x)$.

6. For the two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ below, find $g \circ f$ and $f \circ g$.

$$f(x) = \begin{cases} |2x - 1| & \text{if } x \geq -1 \\ -x(x + 2) & \text{if } x < -1 \end{cases}, \qquad g(x) = \begin{cases} -\sqrt{x - 4} & \text{if } x \geq 4 \\ 1 - x/2 & \text{if } x < 4 \end{cases}$$

Solution:
For $f \circ g$:
For $x \geq 4$ we have

$$-\sqrt{x - 4} \geq -1 \quad \Leftrightarrow \quad x - 4 \leq 1 \quad \Leftrightarrow \quad 4 \leq x \leq 5$$

and for all $x < 4$ we have $1 - \frac{1}{2}x \geq -1$. So $g(x) \geq -1 \Leftrightarrow x \leq 5$ and it follows

$$(f \circ g)(x) = \begin{cases} |2g(x) - 1|, & x \leq 5 \\ -g(x)\big(g(x) + 2\big), & x > 5 \end{cases}$$

$$= \left\{ \begin{array}{ll} |1 - x|, & x < 4 \\ \big| - 2\sqrt{x - 4} - 1\big|, & 4 \leq x \leq 5 \\ 2\sqrt{x - 4} - x + 4, & x > 5 \end{array} \right\} = \begin{cases} |1 - x|, & x < 4 \\ 2\sqrt{x - 4} + 1, & 4 \leq x \leq 5 \\ 2\sqrt{x - 4} - x + 4, & x > 5 \end{cases}$$

For $g \circ f$:
For $x \geq -1$, we have

$$|2x-1| \geq 4 \quad \Leftrightarrow \quad 2x-1 \geq 4 \text{ or } 2x-1 \leq -4 \quad \Leftrightarrow \quad x \geq \frac{5}{2} \qquad (\text{since } x \geq -1)$$

and for $x < -1$ we have $-x(x+2) \geq 4 \Leftrightarrow x^2+2x+4 \leq 0$, which is impossible since the polynomial has no real roots. So $f(x) \geq 4 \Leftrightarrow x \geq \frac{5}{2}$ and therefore

$$(g \circ f)(x) = \begin{cases} -\sqrt{f(x)-4}, & x \geq \frac{5}{2} \\ 1 - \frac{1}{2}f(x), & x < \frac{5}{2} \end{cases}$$

$$= \left\{ \begin{array}{ll} -\sqrt{|2x-1|-4}, & x \geq \frac{5}{2} \\ 1-\frac{1}{2}|2x-1|, & -1 \leq x < \frac{5}{2} \\ \frac{1}{2}x^2+x+1, & x < -1 \end{array} \right\} = \begin{cases} -\sqrt{2x-5}, & x \geq \frac{5}{2} \\ 1-\frac{1}{2}|2x-1|, & -1 \leq x < \frac{5}{2} \\ \frac{1}{2}x^2+x+1, & x < -1 \end{cases}$$

7. State if the following are true or false.

   (a) If f is an even function and g is an odd function, then $h = f \cdot g$ is an odd function.

   (b) If f is an even function and g is an odd function, then $h = f \circ g$ is an odd function.

   (c) A function is either even or odd or both.

Solution:

(a) True. We have

$$h(-x) = f(-x)g(-x) = (f(x)) \cdot (-g(x)) = -f(x)g(x) = -h(x)$$

(b) False. We have

$$h(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = h(x)$$

So h is an even function.

(c) False. Take for example $f(x) = x + x^2$. Then $f(-x) = -x + x^2 \neq f(x) \neq -f(x)$. So this function is neither even nor odd.

8. Calculate the following limits.

   (a) $\lim\limits_{x \to 0} \frac{x^3+4x}{2x}$

   (b) $\lim\limits_{x \to 0} \frac{\sqrt{9+x}-3}{x}$

   (c) $\lim\limits_{x \to 0} \frac{\cos(x)-1}{x^2}$

Solution:

(a)

$$\lim_{x\to 0}\frac{x^3+4x}{2x}=\lim_{x\to 0}\frac{x(x^2+4)}{2x}=\lim_{x\to 0}\frac{x^2+4}{2}=2$$

(b)

$$\lim_{x\to 0}\frac{\sqrt{9+x}-3}{x}=\lim_{x\to 0}\frac{\sqrt{9+x}-3}{x}\cdot\frac{\sqrt{9+x}+3}{\sqrt{9+x}+3}=\lim_{x\to 0}\frac{x}{x(\sqrt{9+x}+3)}=\lim_{x\to 0}\frac{1}{\sqrt{9+x}+3}=\frac{1}{6}$$

(c)

$$\lim_{x\to 0}\frac{\cos(x)-1}{x^2}=\lim_{x\to 0}\frac{-2\sin^2(x/2)}{x^2}=\lim_{x\to 0}(-2)\cdot\frac{\sin(x/2)}{2\cdot x/2}\cdot\frac{\sin(x/2)}{2\cdot x/2}$$
$$=\lim_{x\to 0}(-\frac{1}{2})\cdot\frac{\sin(x/2)}{x/2}\cdot\frac{\sin(x/2)}{x/2}=-\frac{1}{2}$$

9. Calculate the following limits.
   (a) $\lim_{x\to 4}\frac{x^2+5x-36}{x^2-16}$
   (b) $\lim_{x\to 1}\frac{x^n-1}{x-1}$ (*Hint: Try to factorize $x-1$ from the numerator.*)

Solution:

(a)

$$\lim_{x\to 4}\frac{x^2+5x-36}{x^2-16}=\frac{(x-4)(x+9)}{(x-4)(x+4)}=\lim_{x\to 4}\frac{x+9}{x+4}=\frac{13}{8}$$

(b) We use the formula

$$x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$$

to rewrite

$$\begin{aligned}\lim_{x\to 1}\frac{x^n-1}{x-1}&=\lim_{x\to 1}\frac{(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)}{x-1}\\&=\lim_{x\to 1}\left(x^{n-1}+x^{n-2}+\cdots+x+1\right)\\&=n\end{aligned}$$

10. (Multiple choice) The series

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^{500}}{(1.0001)^n}$$

   (a) converges absolutely.
   (b) converges, but not absolutely.
   (c) approaches $+\infty$.
   (d) approaches $-\infty$.

Solution:

(a) is correct. To check absolute convergence we consider the sequence

$$a_n = \left| \frac{(-1)^n n^{500}}{(1.0001)^n} \right| = \frac{n^{500}}{(1.0001)^n}$$

Now we use the ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)^{500}}{(1.0001)^{n+1}}}{\frac{n^{500}}{(1.0001)^n}} = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^{500} \frac{1}{1.0001} = \frac{1}{1.0001} \leq 1$$

This means that the series is absolutely convergent by Alembert's criterion.

11. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sqrt{n}+1} - \frac{n+1}{\sqrt{n+1}+1} \right)$$

   (a) diverges.
   (b) converges to $\frac{1}{2} - \frac{2}{\sqrt{2}+1}$.
   (c) converges to $\frac{1}{2}$.
   (d) converges to 0.

Solution:

We observe that for every $m > 0$, the sequence

$$S_m := \sum_{n=1}^{m} \left(\frac{n}{\sqrt{n}+1} - \frac{n+1}{\sqrt{n+1}+1} \right) = \frac{1}{2} - \frac{m+1}{\sqrt{m+1}+1}$$

approaches $-\infty$. Hence, the series diverges.

12. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n} \right)^n$$

is

   (a) divergent.
   (b) converges to e.
   (c) converges to e^{-1}.
   (d) converges to 1.

Solution:

(a) is correct. Take $a_n = (1 - \frac{1}{n})^n$. We know that

$$\lim_{n\to\infty} \left(1 - \frac{1}{n} \right)^n = e^{-1} \neq 0$$

If the sequence does not converge to zero then the series is divergent.

13. (Multiple choice) The limit

$$\lim_{n\to\infty} \frac{\sqrt[3]{1-\frac{1}{n}}-1}{\sqrt[4]{1-\frac{1}{n}}-1}$$

is

(a) $\frac{3}{4}$
(b) $\frac{4}{3}$
(c) ∞
(d) 0

Solution:
(b) is correct. We use the identity

$$(a^N-1)=(a-1)(a^{N-1}+a^{N-2}+\cdots+1).$$

For $a=\sqrt[3]{1-\frac{1}{n}}$ and $N=3$ we have

$$\left(1-\frac{1}{n}\right)-1=\left(\sqrt[3]{1-\frac{1}{n}}-1\right)\left(\sqrt[3]{\left(1-\frac{1}{n}\right)^2}+\sqrt[3]{1-\frac{1}{n}}+1\right).$$

For $a=\sqrt[4]{1-\frac{1}{n}}$ and $N=4$ we have

$$\left(1-\frac{1}{n}\right)-1=\left(\sqrt[4]{1-\frac{1}{n}}-1\right)\left(\sqrt[4]{\left(1-\frac{1}{n}\right)^3}+\sqrt[4]{\left(1-\frac{1}{n}\right)^2}+\sqrt[4]{1-\frac{1}{n}}+1\right)$$

so

$$\lim_{n\to\infty} \frac{\sqrt[3]{1-\frac{1}{n}}-1}{\sqrt[4]{1-\frac{1}{n}}-1}=\lim_{n\to\infty} \frac{(1-\frac{1}{n})-1}{(1-\frac{1}{n})-1}\cdot\frac{\sqrt[4]{(1-\frac{1}{n})^3}+\sqrt[4]{(1-\frac{1}{n})^2}+\sqrt[4]{1-\frac{1}{n}}+1}{\sqrt[3]{(1-\frac{1}{n})^2}+\sqrt[3]{1-\frac{1}{n}}+1}=\frac{4}{3}$$

14. (Multiple choice) The limit

$$\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n\frac{n+2}{n+1}$$

is

(a) e^2
(b) e
(c) ∞
(d) 0

Solution:

(a) is correct. We have

$$\left(\frac{n+2}{n}\right)^n \frac{n+2}{n+1} = \left(\frac{n+1}{n}\right)^n \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n+1}\right)^{n+1}$$

and

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n+1}\right)^{n+1} = e^2$$

15. (Multiple choice) The limit

$$\lim_{n\to\infty} n^2 \cdot \sin\left(\frac{2n+3}{n^3}\right)$$

is

- (a) 0
- (b) $\frac{1}{2}$
- (c) 2
- (d) ∞

Solution:

(c) is correct. Note that

$$\sqrt{1-x^2} \le \frac{\sin x}{x} \le 1$$

so

$$\sqrt{1-\left(\frac{2n+3}{n^3}\right)^2} \le \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\left(\frac{2n+3}{n^3}\right)} \le 1$$

Since $\lim_{n\to\infty}\left(\frac{2n+3}{n^3}\right) = 0$ then by the squeeze theorem we have that

$$\frac{\sin\left(\frac{2n+3}{n^3}\right)}{\left(\frac{2n+3}{n^3}\right)} = 1.$$

So for the original limit we can write

$$\lim_{n\to\infty} n^2 \cdot \sin\left(\frac{2n+3}{n^3}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\left(\frac{2n+3}{n^3}\right)} \cdot \left(\frac{2n+3}{n^3}\right) \cdot n^2 = \lim_{n\to\infty} \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\left(\frac{2n+3}{n^3}\right)} \cdot \frac{2n^3+3n^2}{n^3} = 2$$

16. Find the values $\alpha \in \mathbb{R}$ such that the limit $\lim_{x\to\alpha} \frac{\tan(x-\alpha)^2}{(x-\alpha)^2}$ exists in $\mathbb{R}$.

Solution:

We observe that $\lim_{x\to\alpha} \frac{\tan(x-\alpha)^2}{(x-\alpha)^2} = \lim_{x\to 0} \frac{\tan(x)^2}{x^2}$. We also know that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. So we have,

$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x\cdot\cos(x)} = \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right) \cdot \left(\lim_{x\to 0} \frac{1}{\cos(x)}\right) = 1$$

Since the second limit exist and is 1. Finally we write

$$\lim_{x\to 0}\frac{\tan(x)^2}{x^2}=\left(\lim_{x\to 0}\frac{\tan(x)}{x}\right)^2=1^2=1$$

So the limit exists for all $\alpha\in\mathbb{R}$.

17. Compute the following limits if they exist.
    (a) $\lim\limits_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$
    (b) $\lim\limits_{x\to a}\frac{\cos(x)-\cos(a)}{x-a}$ with $a\in\mathbb{R}$
    (c) $\lim\limits_{x\to+\infty}(x|\sin(x)|-x^2+4)$

Solution:

(a) Since $1-x^3=(1-x)(1+x+x^2)$, we can use the common divisors of the two denominators and simplify the fractions:

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)&=\lim_{x\to 1}\frac{1+x+x^2-3}{1-x^3}=\lim_{x\to 1}\frac{x^2+x-2}{(1-x)(1+x+x^2)}\\&=\lim_{x\to 1}\frac{(x-1)(x+2)}{(1-x)(1+x+x^2)}=-\lim_{x\to 1}\frac{x+2}{x^2+x+1}=-\frac{3}{3}=-1.\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to a}\frac{\cos(x)-\cos(a)}{x-a}&=\lim_{x\to a}\frac{-2\cdot\sin\left(\frac{x+a}{2}\right)\cdot\sin\left(\frac{x-a}{2}\right)}{x-a}\\&=-\left(\lim_{x\to a}\sin\left(\frac{x+a}{2}\right)\right)\cdot\left(\lim_{x\to a}\frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}\right)=-\sin(a),\end{aligned}$$

as the second limit is 1.

(c)

$$\lim_{x\to+\infty}(x|\sin(x)|-x^2+4)=\left(\lim_{x\to+\infty}x(|\sin(x)|-x)\right)+4=-\infty$$

because $|\sin(x)|$ is bounded so $\lim_{x\to+\infty}|\sin(x)|-x=-\infty$. We have also $\lim_{x\to\infty}x=+\infty$, so $\lim_{x\to+\infty}x(|\sin(x)|-x)=-\infty$. The result follows from the fact that the constant function 4 is bounded.

18. Find the values $\alpha,\beta\in\mathbb{R}$ such that the limit $\lim\limits_{x\to 0}\dfrac{x^2\sin\left(\frac{1}{x}\right)+\alpha|x|}{\sqrt{x^2+\beta\left|\cos\left(\frac{1}{x}\right)\right|}}$ exists in $\mathbb{R}$.

Solution:

We distinguish three cases for β.

1. If $\beta = 0$, the limit is

$$\lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) + \alpha|x|}{|x|} = \lim_{x\to 0} \left(|x|\sin\left(\frac{1}{x}\right) + \alpha\right) = \alpha$$

   Since $0 \le \left|x\sin\left(\frac{1}{x}\right)\right| \le |x|$, the squeeze theorem gives $\lim\limits_{x\to 0} |x|\sin\left(\frac{1}{x}\right) = 0$. So the limit exists for all $\alpha \in \mathbb{R}$ if $\beta = 0$.

2. If $\beta < 0$, $x^2 + \beta\left|\cos\left(\frac{1}{x}\right)\right|$ takes negative values in a neighborhood of $x = 0$. In fact, for $x_n = \frac{1}{2n\pi}$ with $n \in \mathbb{N}^*$ we have

$$\lim_{n\to\infty} \left(x_n^2 + \beta\left|\cos\left(\frac{1}{x_n}\right)\right|\right) = \lim_{n\to\infty} \left(\left(\frac{1}{2n\pi}\right)^2 + \beta\,|\cos(2n\pi)|\right) = \beta < 0.$$

   Thus, the expression is not defined so the limit does not exist.

3. If $\beta > 0$, we need to distinguish whether α is zero or not. If $\alpha = 0$, For all $x \in \mathbb{R}^*$ we have,

$$0 \le \frac{x^2\left|\sin\left(\frac{1}{x}\right)\right|}{\sqrt{x^2 + \beta\left|\cos\left(\frac{1}{x}\right)\right|}} = \frac{|x|\left|\sin\left(\frac{1}{x}\right)\right|}{\sqrt{1 + \frac{\beta}{x^2}\left|\cos\left(\frac{1}{x}\right)\right|}} \le |x|$$

   and so

$$\lim_{x\to 0} \frac{x^2\left|\sin\left(\frac{1}{x}\right)\right|}{\sqrt{x^2 + \beta\left|\cos\left(\frac{1}{x}\right)\right|}} = 0$$

   for all $\beta > 0$.

   If $\alpha \neq 0$, the limit does not exist. In fact, by taking the sequences $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{\frac{\pi}{2} + 2n\pi}$ with $n \in \mathbb{N}^*$, we have

$$\lim_{n\to\infty} \frac{x_n^2 \sin\left(\frac{1}{x_n}\right) + \alpha|x_n|}{\sqrt{x_n^2 + \beta\left|\cos\left(\frac{1}{x_n}\right)\right|}} = \lim_{n\to\infty} \frac{\left(\frac{1}{2n\pi}\right)^2 \sin(2n\pi) + \frac{\alpha}{2n\pi}}{\sqrt{\left(\frac{1}{2n\pi}\right)^2 + \beta\,|\cos(2n\pi)|}} = \lim_{n\to\infty} \frac{\frac{\alpha}{2n\pi}}{\sqrt{\left(\frac{1}{2n\pi}\right)^2 + \beta}} = 0$$

   and

$$\begin{aligned}
\lim_{n\to\infty} \frac{y_n^2 \sin\left(\frac{1}{y_n}\right) + \alpha|y_n|}{\sqrt{y_n^2 + \beta\left|\cos\left(\frac{1}{y_n}\right)\right|}} &= \lim_{n\to\infty} \frac{|y_n|\sin\left(\frac{1}{y_n}\right) + \alpha}{\sqrt{1 + \frac{\beta}{y_n^2}\left|\cos\left(\frac{1}{y_n}\right)\right|}} \\
&= \lim_{n\to\infty} \frac{\left(\frac{\pi}{2} + 2n\pi\right)^{-1} \sin\left(\frac{\pi}{2} + 2n\pi\right) + \alpha}{\sqrt{1 + \beta\left(\frac{\pi}{2} + 2n\pi\right)\left|\cos\left(\frac{\pi}{2} + 2n\pi\right)\right|}} \\
&= \lim_{n\to\infty} \left(\frac{1}{\frac{\pi}{2} + 2n\pi} + \alpha\right) = \alpha
\end{aligned}$$

   as $\cos\left(\frac{\pi}{2} + 2n\pi\right) = 0$ for all $n \in \mathbb{N}^*$. So the limit does not exist (for $\alpha \neq 0$).

All in all, the limit exists if $\beta = 0$ and $\alpha \in \mathbb{R}$ or if $\beta > 0$ and $\alpha = 0$.