Analysis I (English)
Fall Semester 2025–2026

Analysis 1 - Exercise Set 7

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. State if the following statements are true or false. If you think the statement is true, then prove that; otherwise, provide a counterexample.

   (a) If (x_n) is a sequence that converges to 0, then the series $\sum_{n=0}^{\infty} x_n$ converges.

   (b) Let (x_n) and (y_n) be two sequences such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=0}^{\infty} x_n$ diverges, then the series $\sum_{n=0}^{\infty} y_n$ diverges.

   (c) Let (x_n) and (y_n) be sequences such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=0}^{\infty} x_n$ diverges, then the series $\sum_{n=0}^{\infty} y_n$ diverges.

   (d) Let (x_n) and (y_n) be sequences. If the series $\sum_{n=0}^{\infty} x_n$ converges and the sequence (y_n) converges, then the series $\sum_{n=0}^{\infty} x_n y_n$ converges.

Solution:

(a) False. Take $x_n = \frac{1}{n}$ for example.

(b) True. Let $S_n := \sum_{k=0}^{n} x_k$ and $T_n := \sum_{k=0}^{n} y_k$. Then $S_n \leq T_n$ for all $n \in \mathbb{N}$. Since $x_n, y_n \geq 0$ for all $n \in \mathbb{N}$, the sequences (S_n) and (T_n) are both monotone increasing. If $\sum_{n=0}^{\infty} x_n$ diverges, then (S_n) is not bounded above. Since $S_n \leq T_n$ for all $n \in \mathbb{N}$, then also (T_n) is unbounded above, in particular, it cannot converge.

(c) False. Take $x_n = -n$ and $y_n = \frac{1}{n^2}$ for example.

(d) False. Take for example $x_n = y_n = \frac{(-1)^n}{\sqrt{n}}$. Then $\lim_{n\to\infty} y_n = 0$, the series $\sum_{n=0}^{\infty} x_n$ converges by the Leibniz criterion, and the series $\sum_{n=0}^{\infty} x_n y_n = \sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

2. For each of the following, determine whether the series is convergent or divergent.

   (a) $\sum_{n=0}^{\infty} \frac{1}{n^2+n+3}$

   (b) $\sum_{n=0}^{\infty} \frac{2n^2+1}{3n^2+2}$

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

Solution:

(a) Define the sequence $a_n = \frac{1}{n^2+n+3}$ and the sequence $b_n = \frac{1}{n^2}$. We know that for all n we have that $a_n < b_n$. We also know that the series $\sum_{n=0}^{\infty} b_n$ converges. By the comparison criterion (see Exercise 9(b)), we conclude that the series $\sum_{n=0}^{\infty} \frac{1}{n^2+n+3}$ converges.

(b) Define the sequence $a_n = \frac{2n^2+1}{3n^2+2}$, this sequence does not converge to zero. So the series is divergent.

(c) Define the sequence $a_n = \frac{1}{\sqrt{n+3}}$. This sequence converges to zero and is monotone. By the Leibniz criterion alternating series we can then directly conclude that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ converge.

3. For each of the following, determine whether the series is convergent or divergent.

   (a) $\sum_{n=0}^{\infty} \frac{\sin\left(2n^2\right)}{n^2+3}$

   (b) $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

   (c) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n+3}$

Solution:

(a) To figure this out define the new series by

$$\sum_{n=0}^{\infty} \left| \frac{\sin\left(2n^2\right)}{n^2+3} \right|$$

We know that absolute converge implies convergence. So if the above series converges, then the original series $\sum_{n=0}^{\infty} \frac{\sin\left(2n^2\right)}{n^2+3}$ must also converge. Define the sequence $a_n = \left| \frac{\sin\left(2n^2\right)}{n^2+3} \right|$ and the sequence $b_n = \frac{1}{n^2}$. We know that $a_n < b_n$ for all $n \geq 1$. We also know that the series $\sum_{n=0}^{\infty} b_n$ converges; by the comparison criterion (see Exercise 9(b)), we conclude that the series $\sum_{n=0}^{\infty} \frac{\sin\left(2n^2\right)}{n^2+3}$ converge.

(b) Applying the ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{10}{16} < 1$$

So this series is convergent by Alembert's criterion.

(c) Let $x_n := (-1)^n \frac{n}{n+3}$. The series diverges. Indeed, if it were convergent, then (x_n) would be convergent and have limit 0. Then also $\lim_{n\to\infty} |x_n|$ would be 0. This is in contradiction with

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} \frac{n}{n+3} = 1 \neq 0.$$

4. For each of the following, determine whether the series is convergent or divergent.

   (a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+3)}}$

   (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^5}}{n^3+1}$

   (c) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$

Solution:

(a) We observe that

$$0 \leq \frac{1}{\sqrt{n(n^2+3)}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{\frac{3}{2}}}.$$

Therefore, the series is convergent by the comparison criterion (see Exercise 9(b)), because the series $\sum_{k=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges as $\frac{3}{2} > 1$.

(b) We observe that

$$\frac{\sqrt{n^5}}{n^3+1} = \frac{1}{n^{\frac{1}{2}} + \frac{1}{n^{\frac{5}{2}}}}$$

is very close to $\frac{1}{n^{\frac{1}{2}}}$, so we expect that the series does not converge, because the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges. We also observe that $\frac{1}{n^{\frac{1}{2}}} \geq \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence we try to use the comparison criterion for the series we are studying. We observe that

$$\frac{1}{n^{\frac{1}{2}} + \frac{1}{n^{\frac{5}{2}}}} \geq \frac{1}{n} \tag{1}$$

holds if and only if

$$\frac{1}{n^{\frac{1}{2}}} + \frac{1}{n^{\frac{7}{2}}} \leq 1. \tag{2}$$

Since the sequence $\left(\frac{1}{n^{\frac{1}{2}}} + \frac{1}{n^{\frac{7}{2}}}\right)$ converges to 0, there exists $N > 0$ such that (2) holds for all $n > N$. Then also (1) holds for all $n > N$. Hence, we conclude that the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^5}}{n^3+1}$ diverges by the comparison criterion (see Exercise 9(b)), because the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(c) We observe that

$$0 \leq \frac{(k!)^2}{(2k)!} = \frac{k \cdot (k-1) \cdots 1}{2k \cdot (2k-1) \cdots (k+1)} = \prod_{j=0}^{k-1} \frac{k-j}{2k-j} = \prod_{j=0}^{k-1} \frac{1}{1 + \frac{k}{k-j}} \leq \left(\frac{1}{2}\right)^k$$

because $\frac{k}{k-j} \geq 1$. Therefore, the series is convergent by the comparison criterion (see Exercise 9(b)), because the series $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ converges.

5. (Multiple choice) The series

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$$

is

(a) divergent.

(b) converges to $2 + \sqrt{2}$.

(c) converges to $2 - \sqrt{2}$.

(d) cannot be determined.

Solution:
(b) is correct. We use the identity

$$a^N - 1 = (a-1)(a^{N-1} + a^{N-2} + \cdots + 1)$$

to see that

$$S_N := \sum_{n=0}^{N} \left(\frac{1}{\sqrt{2}}\right)^n = \frac{\left(\frac{1}{\sqrt{2}}\right)^{N+1} - 1}{\frac{1}{\sqrt{2}} - 1}$$

So

$$\lim_{N\to\infty} S_N = \frac{1}{1 - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \sqrt{2}(\sqrt{2}+1) = 2 + \sqrt{2}.$$

6. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

   (a) converges absolutely.
   (b) converges, but not absolutely.
   (c) diverges to $+\infty$.
   (d) diverges to $-\infty$.

Solution:
(b) is correct. The point of this problem is really just to acknowledge that it is in fact an alternating series. To see this we need to acknowledge that,

$$\cos(n\pi) = (-1)^n$$

and so the series is really,

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Now define a sequence $b_n = \frac{1}{\sqrt{n}}$. The series is convergent, if $\lim_{n\to\infty} b_n = 0$ and b_n is monotone. We see that indeed:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

and also b_n is a decreasing sequence. But on the other hand

$$\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi)}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = +\infty$$

So the series is convergent, but not absolutely.

7. Terminate the proof that we started in class showing the convergence of $\sum_{i=1}^{\infty} \frac{(-1)^i}{i}$. This is what we have proven in class and that you can assume:

(a) the subsequence (y_k) of (s_n),

$$y_k := s_{2k+1} = \sum_{i=0}^{2k+1} \frac{(-1)^i}{i}$$

is strictly increasing;

(b) (y_k) is bounded; in particular (y_k) converges to a limit $y \in \mathbb{R}$.

(*H*int: Show that (s_n) is a Cauchy sequence. Use the fact that since (y_k) converges, then it is Cauchy, and that $s_{2k} - \frac{1}{2k+1} = s_{2k+1}$.)

Solution:

Since (y_k) converges, in particular it is a Cauchy sequence. Hence for any $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that for all $k_1, k_2 \in \mathbb{N}$, $k_1, k_2 \geq k_\epsilon$,

$$|y_{k_1} - y_{k_2}| < \epsilon.$$

But this is equivalent to saying that for any $\epsilon > 0$ there exists $n'_\epsilon \in \mathbb{N}$ such that for all $n'_1, n'_2 \in \mathbb{N}$, n'_1, n'_2 **odd**, $n'_1, n'_2 \geq n'_\epsilon$,

$$|s_{n'_1} - s_{n'_2}| < \epsilon.$$

We have to find n_ϵ such that

$$|s_{n_1} - s_{n_2}| < \epsilon.$$

holds for any $n_1, n_2 \geq n_\epsilon$ and not just the odd ones, for any fixed value of ϵ.
If n_1 is even and n_2 is odd, then we can write

$$|s_{n_1} - s_{n_2}| = |s_{n_1 - 1} + \frac{1}{n_1} - s_{n_2}| \leq \frac{1}{n_1} + |s_{n_1 - 1} - s_{n_2}|,$$

where the inequality follows from the triangle inequality. If n_1, n_2 are even, then we can write

$$|s_{n_1} - s_{n_2}| = |s_{n_1 - 1} + \frac{1}{n_1} - s_{n_2 - 1} + \frac{1}{n_2}| \leq \frac{1}{n_1} + \frac{1}{n_2} + |s_{n_1 - 1} - s_{n_2 - 1}|,$$

where the inequality follows from the triangle inequality.
Fix $\epsilon > 0$. Let $n''_\epsilon \in \mathbb{N}$ be such that $\frac{1}{n''_\epsilon} < \frac{\epsilon}{4}$. Let $n'''_\epsilon := n'_{\frac{\epsilon}{2}}$ be the index in the definition of Cauchy sequence for (y_k) with $\frac{\epsilon}{2}$. Define $n_\epsilon = \max n''_\epsilon, n'''_\epsilon$. Then for natural numbers $n, m, r, t \geq n_\epsilon$, r, t odd, we have that

$$\frac{1}{n}, \frac{1}{m} \leq \frac{\epsilon}{4},$$
$$|s_r - s_t| < \frac{\epsilon}{2}.$$

If n_1 is even and n_2 is odd, $n_1, n_2 \geq n_\epsilon$ then we can write

$$|s_{n_1} - s_{n_2}| = |s_{n_1 - 1} + \frac{1}{n_1} - s_{n_2}| \leq \frac{1}{n_1} + |s_{n_1 - 1} - s_{n_2}| < \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon.$$

If n_1, n_2 are even, then we can write

$$|s_{n_1} - s_{n_2}| = |s_{n_1 - 1} + \frac{1}{n_1} - s_{n_2 - 1} + \frac{1}{n_2}| \leq \frac{1}{n_1} + \frac{1}{n_2} + |s_{n_1 - 1} - s_{n_2 - 1}| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$

8. (a) Show that for every $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k=1}^{n}(-1)^{k+1}\frac{k}{k^2-\frac{1}{4}} = 1 + \frac{(-1)^{n+1}}{2n+1}.$$

(b) Use the previous part to compute $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{k^2-\frac{1}{4}}$.

(c) Is the series absolutely convergent?

Solution:

We observe that

$$b_n := \sqrt[n]{\left|\frac{a^n}{a^{2n}-1}\right|} = \frac{a}{\sqrt[n]{|a^{2n}-1|}} = \frac{1}{a\sqrt[n]{|1-(a^{-1})^{2n}|}}$$

If $a < 1$, then $\frac{a}{\sqrt[n]{|a^{2n}-1|}} = \frac{a}{\sqrt[n]{1-a^{2n}}}$. Moreover, the sequence a^{2n} decreases, then the sequence $1-a^{2n}$ increases. So

$$\sqrt[n]{1-a} \le \sqrt[n]{1-a^{2n}} \le 1$$

for all $n \ge 1$. Since $(\sqrt[n]{1-a})$ converges with limit 1, the sequence $(\sqrt[n]{1-a^{2n}})$ converges with limit 1 by squeeze theorem. Then (b_n) converges with limit a. Since $a < 1$ we conclude that the series converges by the Cauchy criterion.

If $a > 1$, then $a^{-1} < 1$ and $\frac{1}{a\sqrt[n]{|1-(a^{-1})^{2n}|}} = \frac{a^{-1}}{\sqrt[n]{1-(a^{-1})^{2n}}}$. Repeating the previous argument with a replaced by a^{-1}, we obtain that (b_n) converges with limit a^{-1}. Since $a^{-1} < 1$ we conclude that the series converges by the Cauchy criterion.

9. Compute $\lim\limits_{n\to\infty} \sqrt[n]{n!}$

Solution:

First note that $n! \ge n(n-1)(n-2)\cdots\lfloor\frac{n}{2}\rfloor \ge \lfloor\frac{n}{2}\rfloor^{\lfloor\frac{n}{2}\rfloor}$. Thus, $\sqrt[n]{n!} \ge \lfloor\frac{n}{2}\rfloor^{\lfloor\frac{n}{2}\rfloor\frac{1}{n}}$. We can also see that $\lfloor\frac{n}{2}\rfloor\frac{1}{n} \ge \frac{1}{10}$ for sufficiently large n. Thus, $\sqrt[n]{n!} \ge \lfloor\frac{n}{2}\rfloor^{\lfloor\frac{n}{2}\rfloor\frac{1}{n}} \ge \lfloor\frac{n}{2}\rfloor^{\frac{1}{10}}$ and the latter diverges to $+\infty$. Hence, $\lim\limits_{n\to\infty}\sqrt[n]{n!} = +\infty$.

10. Let (x_n) be a sequence.

(a) Show that if $\lim\limits_{n\to\infty}\sqrt[n]{|x_n|} = \rho$, with $\rho > 1$, then (x_n) is unbounded. In particular, it diverges.

(b) Show that if $\lim\limits_{n\to\infty}\sqrt[n]{|x_n|} = \rho$, with $0 \le \rho < 1$, then (x_n) converges to 0. In particular, it is bounded.

(c) Provide two sequences (y_n) and (z_n) with the following properties: (y_n) converges and $\lim\limits_{n\to\infty}\sqrt[n]{|y_n|} = 1$, and (z_n) diverges to $+\infty$ and $\lim\limits_{n\to\infty}\sqrt[n]{|z_n|} = 1$

Solution:

(a) Note that we have $\lim_{n\to\infty} \frac{\sqrt[n]{|x_n|}}{\rho} = 1$.

Suppose that (x_n) is bounded, say $|x_n| \leq C \quad \forall n \in \mathbb{N}$ for some positive constant C. Thus,

$$1 = \lim_{n\to\infty} \frac{\sqrt[n]{|x_n|}}{\rho} \leq \lim_{n\to\infty} \frac{\sqrt[n]{C}}{\rho} = \frac{1}{\rho} < 1$$

since $\rho > 1$. This is a contradiction and therefore (x_n) is unbounded and it must diverge.

(b) Note that we have $\lim_{n\to\infty} \sqrt[n]{|x_n|} = \rho$ with $0 \leq \rho < 1$.

Let $\epsilon = \frac{1-\rho}{2}$. Then, by definition of convergence, there exists $N \in \mathbb{N}$ such that, for evey $n \geq N$, $|\sqrt[n]{|x_n|} - \rho| \leq \frac{1-\rho}{2}$. In particular, for every $n \geq N$, we have $0 \leq |\sqrt[n]{|x_n|} \leq \frac{1+\rho}{2}$. Also, notice that $\frac{1+\rho}{2} < 1$. Then, by raising the inequality to the n-th power, we have

$$0 \leq |x_n| \leq \left(\frac{1+\rho}{2}\right)^n.$$

In turn, we have

$$-\left(\frac{1+\rho}{2}\right)^n \leq x_n \leq \left(\frac{1+\rho}{2}\right)^n$$

for every $n \geq N$. Since $0 < \frac{1+\rho}{2} < 1$, its geometric sequence converges to 0. Then, by the squeeze theorem, also (x_n) converges to 0.

(c) Consider the constant sequence $y_n = 5$ for all n and the sequence $z_n = n$.

11. Let $(t_n) \subset \mathbb{R}^*$ be a sequence. Assume that $\lim_{n\to\infty} t_n = 0$. Show that $\lim_{n\to\infty} \frac{\sin(t_n)}{t_n} = 1$.
(*Hint: recall that for* $x \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} 0 \leq \sin(x) \leq x \leq \tan(x) &\Rightarrow 1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)} \Rightarrow \cos(x) \leq \frac{\sin(x)}{x} \leq 1 \\ &\Rightarrow \cos(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \Rightarrow 1 - \sin(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \\ &\Rightarrow 1 - x^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \Rightarrow \sqrt{1-x^2} \leq \frac{\sin(x)}{x} \leq 1.) \end{aligned}$$

Solution:

As $\lim_{n\to\infty} t_n = 0$, then there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, then $|t_n| \leq \frac{\pi}{2}$. Moreover, as $\sin(x)$ is an odd function, then

$$\frac{\sin(t_n)}{t_n} = \frac{\sin(|t_n|)}{|t_n|}.$$

Then, by the hint, $\forall n \geq N$,

$$\sqrt{1-t_n^2} \leq \frac{\sin(|t_n|)}{|t_n|} = \frac{\sin(t_n)}{t_n} \leq 1.$$

As $\lim_{n\to\infty} t_n = 0$, then

$$\lim_{n\to\infty} \sqrt{1-t_n^2} = 1.$$

[Prove this!! *Hint: for* $0 < x < 1$, $1 > \sqrt{1-x^2} > 1 - x^2 > 1 - x$.]
Hence, the squeeze theorem for sequences implies the desired conclusion.

12. Show that if $\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = 1$, then anything can happen for $\sum_{n=0}^{\infty} x_n$. That is, it is possible to find sequences (x_n) such that:

    (a) (x_n) is unbounded;

    (b) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ diverges;

    (c) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ converges absolutely;

    (d) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ converges but not absolutely.

    For each item above, provide an example.

Solution:

(a) Take $x_n = \log(n)$.

(b) Take $x_n = \frac{1}{n}$.

(c) Take $x_n = \frac{1}{n^2}$.

(d) Take $x_n = (-1)^n \frac{1}{n}$.

13. For each of the following, determine whether the series is convergent or divergent.

    (a) $\sum_{k=1}^{\infty} \frac{\pi^k}{k\cdot 2^k}$

    (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+4}-\sqrt{n+2}}{n}$

    (c) $\sum_{k=2}^{\infty} \frac{k^2-1}{(k-1)^3}$

Solution:

(a) We observe that

$$\frac{\pi^k}{k\cdot 2^k} = \left(\frac{\pi}{2}\right)^k \cdot \frac{1}{k} \geq \frac{1}{k} \geq 0$$

for all $k \geq 1$. So the series diverges by the comparison criterion, because the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

(b) We observe that

$$0 \leq \frac{\sqrt{n+4}-\sqrt{n+2}}{n} = \frac{n+4-(n+2)}{n(\sqrt{n+4}+\sqrt{n+2})} = \frac{2}{n(\sqrt{n+4}+\sqrt{n+2})} \leq \frac{2}{n(\sqrt{n}+\sqrt{n})} = \frac{1}{n^{\frac{3}{2}}}.$$

Therefore, the series is convergent by the comparison criterion, because the series $\sum_{k=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges as $\frac{3}{2} > 1$.

(c) We observe that

$$\frac{k^2-1}{(k-1)^3} = \frac{k+1}{(k-1)^2} \geq \frac{1}{k} \geq 0.$$

So the series diverges by the comparison criterion, because the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.