Analysis I (English)
Roberto Svaldi and Stefano Filipazzi
Fall Semester 2021–2022

Analysis 1 - Exercise Set 6

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Compute, if they exists, the limits of the following sequences

   (a) $\sqrt[n]{\frac{3}{n}}$

   (b) $(-1)^n \left(\frac{n^2+1}{n-1}\right)$

   (c) $\frac{1}{n^2}\left(\sqrt{1+n+\pi n^2+\frac{\sin(n)}{n}}-1\right)$

   (d) $\sqrt[n]{n\log(n)}$ (*Hint*: $1 < \log(n) < n$ for $n > 3$)

   (e) $n^2\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1\right)$

   (f) $\left(\frac{n-1}{n}\right)^{n^2}$

   (g) $\sqrt[n]{\frac{2n}{3n^2-1}}$

   (h) $\frac{4n^2-2\pi}{-n^3+\sqrt{7}n}$

   (i) $\frac{(n+1)!}{n!-(n+1)!}$

   (j) $\frac{\sqrt{\frac{cos(n)}{n^2}+1}-1}{\sqrt{e-\frac{1}{n}}-\sqrt{e}}$

Solution:

(a)

$$\lim_{n\to\infty}\sqrt[n]{\frac{3}{n}} = \frac{\lim_{n\to\infty}\sqrt[n]{3}}{\lim_{n\to\infty}\sqrt[n]{n}} = \frac{1}{1} = 1$$

(b) Let $a_n = (-1)^n\left(\frac{n^2+1}{n-1}\right)$. Remark that the sequence

$$x_n = |a_n| = \frac{n^2+1}{n-1} = \frac{1+n^{-2}}{n^{-1}-n^{-2}}$$

approaches $+\infty$. Then (a_n) is divergent. The subsequence $a_{2n} = x_{2n}$ converges to $+\infty$; on the other hand, the subsequence $a_{2n+1} = -x_{2n+1}$ converges to $-\infty$. We conclude that a_n does not admit a limit.

(c) We have

$$\frac{1}{n^2}\left(\sqrt{1+n+\pi n^2+\frac{\sin(n)}{n}}-1\right)=\sqrt{\frac{1}{n^4}+\frac{1}{n^3}+\frac{\pi}{n^2}+\frac{\sin(n)}{n^5}}-\frac{1}{n^2}$$

so the limit is zero. (Remark that $\lim_{n\to\infty}\frac{\sin(n)}{n^5}=\lim_{n\to\infty}\frac{\sin(n)}{n}\frac{n}{n^5}$ is zero, because it is the limit of the product of a bounded sequences with a sequences that converges to 0.)

(d) Using the hint, we have that

$$\sqrt[n]{n}<\sqrt[n]{n\log(n)}<\sqrt[n]{n^2}.$$

Applying the squeeze theorem, we get that the limit is 1.

(e) The issue is that $\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1$ convergence to zero, but we do not know the speed of convergence, so we do not know how to compare it with n^2. To this end, we kill the square root as usual writing

$$n^2\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1\right)=\frac{n^2\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1\right)\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}+1\right)}{\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}+1\right)}=$$

$$=\frac{n^2\left(\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}\right)}{\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}+1\right)}=\frac{n+\pi+\frac{\sin(n)}{n^3}}{\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}+1\right)}.$$

We conclude that the limit is $+\infty$, because the denominator is a convergent sequence with limit $\neq 0$, the numerator is the sum of a sequence (n) that approaches $+\infty$ with a sequence $(\pi+\frac{\sin(n)}{n^3})$ that is bounded below.

(f) We have

$$\left(\frac{n-1}{n}\right)^{n^2}=\left(\left(1-\frac{1}{n}\right)^n\right)^n$$

We know that $\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)^n\right)=e^{-1}$. Hence we expect that $\left(\frac{n-1}{n}\right)^{n^2}$ converges to 0. Let's prove it.

Let $\varepsilon>0$ such that $0<\varepsilon<\min\{e^{-1},1-e^{-1}\}$. Let $a=e^{-1}-\varepsilon$ and $b=e^{-1}+\varepsilon$. Then $0<a<e^{-1}<b<1$.

Since $\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)^n\right)=e^{-1}$, there exists an N such that for all $n>N$ we have

$$\left|\left(1-\frac{1}{n}\right)^n-e^{-1}\right|<\varepsilon,$$

which is equivalent to

$$a<\left(\frac{n-1}{n}\right)^n<b.$$

Then

$$a^n<\left(\frac{n-1}{n}\right)^{n^2}<b^n$$

and we conclude applying squeeze theorem.

(g) We have

$$\sqrt[n]{\frac{2n}{3n^2-1}} = \frac{1}{\sqrt[n]{n}}\sqrt[n]{\frac{2n^2}{3n^2-1}} = \frac{1}{\sqrt[n]{n}}\sqrt[n]{\frac{2}{3-\frac{1}{n^2}}}$$

both factors converge now to 1, so the limit is 1.

(h) The limit is 0.

(i) Dividing numerator and denominator by $n!$ we get that

$$\frac{(n+1)!}{n!-(n+1)!} = -\frac{n+1}{n}$$

so the limit is -1.

(j) Numerator and denominator both tend to zero, and we tackle the square root in the usual way, our sequence is indeed equal to

$$\frac{\left(\frac{\cos(n)}{n^2}+1-1\right)\left(\sqrt{e+\frac{1}{n}}+\sqrt{e}\right)}{\left(\sqrt{e}-\frac{1}{n}-\sqrt{e}\right)\left(\sqrt{\frac{\cos(n)}{n^2}+1}+1\right)}$$

Which in turn equals

$$\left(\frac{\cos(n)}{n}\right)\left(\frac{\sqrt{e+\frac{1}{n}}+\sqrt{e}}{\sqrt{\frac{\cos(n)}{n^2}+1}+1}\right)$$

the first factor goes to zero, the second converge to a real number (actually $\sqrt{e}$, but this does not matter), so the limit is zero.

2. Let $a, b \in \mathbb{R}_+$ and (x_n) be a sequence defined by the recurrence relation

$$x_{n+1} = ax_n^2 \quad x_0 = b.$$

(a) Show by induction that every element in the sequence (x_n) is given by

$$x_n = a^{2^n-1}b^{2^n}.$$

(b) Use part (a) to compute

$$\lim_{n\to+\infty} x_n.$$

Solution:

(a) For $n = 0$ we have

$$x_0 = a^{2^0-1}b^{2^0} = b$$

which is true. Assuming that $x_n = a^{2^n-1}b^{2^n}$ for some n, we have

$$\begin{aligned} x_{n+1} =& ax_n^2 \\ =& a \cdot \left(a^{2^n-1}b^{2^n}\right)^2 \\ =& a \cdot a^{2\cdot 2^n-2}b^{2\cdot 2^n} \\ =& a^{2^{n+1}-1}b^{2^{n+1}}. \end{aligned}$$

(b) To calculate the limit, we first rewrite x_n as

$$x_n = a^{2^n-1}b^{2^n} = \frac{(ab)^{2^n}}{a}$$

Now we have 3 different possibilities. If $ab = 1$ then

$$\lim_{n\to+\infty} x_n = \frac{(ab)^{2^n}}{a} = \frac{1}{a}$$

if $ab < 1$ then

$$\lim_{n\to+\infty} x_n = \frac{(ab)^{2^n}}{a} = 0$$

if $ab > 1$ then

$$\lim_{n\to+\infty} x_n = \frac{(ab)^{2^n}}{a} = +\infty$$

3. Show that the following recursive sequence is convergent and calculate the limit

$$a_n = \frac{7}{3} - \frac{1}{1+a_{n-1}}, \quad a_1 = 1.$$

Solution:

If the limit $\lim_{n\to\infty} a_n = a$ exists, it should satisfy the equation

$$a = \frac{7}{3} - \frac{1}{1+a} \quad \Leftrightarrow \quad 1 = \left(\frac{7}{3} - a\right)(1+a) \quad \Leftrightarrow \quad 0 = \frac{4}{3} + \frac{4}{3}a - a^2 \quad \Leftrightarrow$$
$$3a^2 - 4a - 4 = (3a+2)(a-2) = 0 \quad \Leftrightarrow \quad a = 2 \text{ or } a = -\frac{2}{3}.$$

We show by induction that $a_n \geq 0$ for all $n \in \mathbb{N}^*$. We have $a_1 = 1 \geq 0$. If $a_{n-1} \geq 0$, then

$$a_n = \frac{7}{3} - \frac{1}{1+a_{n-1}} \geq \frac{7}{3} - 1 = \frac{4}{3} \geq 0.$$

So the only possible limit for (a_n) is $a = 2$.

We also show (by induction) that 2 is an upper bound for (a_n). We have $a_1 = 1 \leq a$. If $0 \leq a_{n-1} \leq a$, we than have

$$a_n = \frac{7}{3} - \frac{1}{1+a_{n-1}} \leq \frac{7}{3} - \frac{1}{1+a} = 2 = a\,.$$

Showing that (a_n) is an increasing sequence: for $n \geq 2$ we have

$$a_n - a_{n-1} = \frac{7}{3} - \frac{1}{1+a_{n-1}} - a_{n-1} = \frac{4 + 4a_{n-1} - 3a_{n-1}^2}{3(1+a_{n-1})} \geq 0$$

if and only if $4 + 4a_{n-1} - 3a_{n-1}^2 \geq 0$. The last inequality is true because $0 \leq a_{n-1} \leq 2$. Since $(a_n)_{n\geq1}$ is increasing an bounded sequence then it is convergent with the limit $a = 2$.

4. This question is going to show that, whenever we have a sequence that is defined recursively,

we need to show that it converges, and that computing the candidates for the limit is not enough.

Consider the sequence defined as $a_1 = 10$, $a_{n+1} = a_n^2$ for $n \geq 1$.

(a) Show that, if the limit of (a_n) exists, then it is either 0 or 1.

(b) Show that (a_n) diverges to $+\infty$.

Solution:

(a) Assume that (a_n) converges, and let L denote its limit. Then, by what we saw in worksheet 5, also (a_{n+1}) converges to L. So, by the recursive relation, we have $L = L^2$. Then, the solutions to the equation $L^2 - L = 0$ are 0 and 1, showing the claim.

(b) We can use induction to show that $a_n = 10^{2^{n-1}}$. As it is well known that the geometric sequence (10^m) diverges to $+\infty$, then so does $(10^{2^{n-1}})$, since it can be regarded as a subsequence of (10^m).

5. Compute the limit of $a_n = \left(\frac{n+3}{n+1}\right)^n$ using subsequences. (*Hint: first, manipulate the definition of a_n so that it looks more to the sequence of a previous exercise, then use the subsequence with odd indices.*)

Solution:
Write

$$a_n = \left(\frac{n+3}{n+1}\right)^n = \left(1 + \frac{2}{n+1}\right)^n$$

so that

$$a_{2n+1} = \left(1 + \frac{1}{n+1}\right)^{2n+1} = \left(\left(1 + \frac{1}{n+1}\right)^{n+1}\right)^2 \left(1 + \frac{1}{n+1}\right)^{-1}$$

so a_{2n+1} converges to e^2. Now, we argue as in Exercise 17 in the previous sheet.

6. State if the following statements are true or false. If you think the statement is true, then prove that; otherwise, provide a counterexample.

(a) If a sequence is not bounded above, it must be increasing.

(b) Any monotone sequence has a convergent subsequence.

(c) If (a_n) has no divergent subsequence, then (a_n) is convergent.

(d) If (a_n) is Cauchy convergent, then also $(|a_n|)$ is Cauchy convergent.

(e) If (a_n) is a Cauchy sequence, then the sequence $b_n = c \cdot a_n$, $c \neq 0$ is a Cauchy sequence.

(f) If (a_n) is Cauchy, there exists $\varepsilon > 0$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \in \mathbb{N}$.

(g) Any sequence has a convergent subsequence.

(h) If (a_n) and (b_n) are Cauchy sequences, then the sequence $c_n = a_n + b_n$ is a Cauchy sequence.

Solution:

(a) False. Take $a_n = 0$ if n is even and $a_n = n$ if n is odd.

(b) False. Take $a_n = n$.

(c) True. (a_n) is a subsequence of (a_n) itself. So by definition it should be convergent.

(d) True. It follows directly from the inequality:

$$\big||a_n| - |a_m|\big| \leq |a_n - a_m|.$$

(e) True. A shortcut is to use that fact that, for a sequence of real numbers, being Cauchy is equivalent to be convergent, and the statement is of course true if we replace Cauchy with convergent. Let us now give the proof using just the definition of Cauchy sequence.

To show that (b_n) is Cauchy we must show that for any $\epsilon > 0$, there is N such that for all $i, j > N$, $|b_i - b_j| < \epsilon$. Let $\epsilon > 0$ be given. Since (a_n) is Cauchy, there is N such that for all $i, j > N$ we have

$$|a_i - a_j| < \frac{\epsilon}{|c|}$$

Now for all $i, j > N$ we have

$$|b_i - b_j| = |ca_i - ca_j| = |c| \cdot |a_i - a_j| \leq |c| \cdot \frac{\epsilon}{|c|} = \epsilon.$$

So (b_n) is Cauchy.

(f) True. A sequence is Cauchy if and only if it is convergent. So (a_n) is a convergent sequence, let $a \in \mathbb{R}$ be its limit. Then we can find $C \in \mathbb{R}$ such that

$$|a_n - a| \leq C$$

for all $n \in \mathbb{N}$. Using the triangle inequality we have

$$|a_m - a_n| \leq |a_m - a| + |a - a_n| \leq 2C$$

for all $m, n \in \mathbb{N}$. So it is enough to take $\varepsilon = 2C$.

(g) False. Take $a_n = n$.

(h) True. Since (a_n) and (b_n) are Cauchy sequences they must converge. By Algebra of Limits the sequence $c_n = a_n + b_n$ must converge, which is equivalent to c_n being a Cauchy sequence.

7. Show if the sequence

$$a_n = \frac{\sin(a_{n-1}) + 1}{2} \qquad a_1 = 0$$

satisfies the definition of Cauchy sequence. (*Hint: Use the trigonometric formulas from Exercise Sheet 1*)

Solution:

We have for all integers $n \geq 2$:

$$|a_{n+1} - a_n| = \frac{1}{2}\left|\sin(a_n) - \sin(a_{n-1})\right| = \frac{1}{2}\left|2\sin\left(\frac{a_n - a_{n-1}}{2}\right)\cos\left(\frac{a_n + a_{n-1}}{2}\right)\right|$$
$$\leq \left|\sin\left(\frac{a_n - a_{n-1}}{2}\right)\right| \leq \frac{|a_n - a_{n-1}|}{2} \,,$$

In the last inequality we use the fact that $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$.

If we apply the above inequality $n-1$ times, we obtain

$$|a_{n+1} - a_n| \leq \frac{|a_2 - a_1|}{2^{n-1}} = \frac{\left|\frac{1}{2} - 0\right|}{2^{n-1}} = \frac{1}{2^n} \,.$$

For all couple of integers $n > m \geq 2$, by using the triangle inequality we obtain:

$$|a_n - a_m| \leq \sum_{k=m}^{n-1} |a_{k+1} - a_k| \,,$$

so

$$|a_n - a_m| \leq \sum_{k=m}^{n-1} \frac{1}{2^k} = \frac{1}{2^m} \sum_{k=0}^{n-m-1} \left(\frac{1}{2}\right)^k$$
$$= \frac{1}{2^m} \frac{1 - \left(\frac{1}{2}\right)^{n-m}}{1 - \frac{1}{2}} = \frac{1}{2^{m-1}}\underbrace{\left(1 - \left(\frac{1}{2}\right)^{n-m}\right)}_{\leq 1} \leq \frac{1}{2^{m-1}} \,.$$

Since the sequence $(\frac{1}{2^{m-1}})$ converges to 0, for every $\varepsilon > 0$ there exists $M_\varepsilon > 2$ such that for all $m \geq M_\varepsilon$ we have $\frac{1}{2^{m-1}} < \varepsilon$. Then $|a_n - a_m| \leq \varepsilon$ for all $n > m \geq M_\varepsilon$. We conclude that $|a_n - a_m| \leq \varepsilon$ for all $n, m \geq M_\varepsilon$. Therefore it follows that $(a_n)_{n\geq 1}$ is a Cauchy sequence.

8. Let (a_n) and (b_n) be two sequences. Show the following facts.

   (a) Assume that (a_n) and (b_n) are bounded. Prove that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

   (b) Provide an example of sequences (a_n) and (b_n) such that the inequality in part (a) is strict.

   (c) Assume that $\liminf a_n = 5$. Show that there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, $a_n \geq 4$.

   (d) Assume (b_n) is defined as follows:

$$b_n = \begin{cases} \frac{100}{n} \text{ if } 3|n \\ 2 - \frac{1}{n} \text{ if } 3|n-1 \\ \frac{1}{2} \text{ if } 3|n-2 \end{cases}$$

   Compute $\limsup b_n$, $\liminf b_n$, and exhibit a subsequence of (b_n) converging to $\limsup b_n$ and a subsequence converging to $\liminf b_n$.

Solution:

(a) We have that $\limsup a_n = \lim_{n\to\infty} \sup\{a_k | k \geq n\}$, $\limsup b_n = \lim_{n\to\infty} \sup\{b_k | k \geq n\}$, and $\limsup(a_n + b_n) = \lim_{n\to\infty} \sup\{a_k + b_k | k \geq n\}$ and all of these limits exist finite, as (a_n), (b_n), and therefore $(a_n + b_n)$ are bounded.

Fix $n \in \mathbb{N}$ and $h \geq n$. Then, $a_h + b_h \leq \sup\{a_k|k \geq n\} + b_h \leq \sup\{a_k|k \geq n\} + \sup\{b_k|k \geq n\}$, where we used the definition of sup and the fact that $h \geq n$. Thus, $\sup\{a_k|k \geq n\}+\sup\{b_k|k \geq n\}$ is an upper bound for $\{a_k+b_k|k \geq n\}$. As sup is the least upper bound of a set, it follows that $\sup\{a_k + b_k|k \geq n\} \leq \sup\{a_k|k \geq n\} + \sup\{b_k|k \geq n\}$.

(b) Consider $(a_n) = (-1)^n$ and $(b_n) = (-1)^{n+1}$. Then, $\limsup a_n = \limsup b_n = 1$, but $a_n + b_n = 0$ for all n. Thus, $\limsup(a_n + b_n) = 0 < 2 = \limsup a_n + \limsup b_n$.

(c) By definition of lim inf, we have $\lim_{n\to\infty} \inf\{a_k|k \geq n\} = 5$. Now, we apply the definition of convergence to the sequence $(\inf\{a_k|k \geq n\})$ with limit 5 and $\epsilon = 1$. Thus, there exist $N \in \mathbb{N}$ such that, for every $n \geq N$, we have $|\inf\{a_k|k \geq n\}-5| \leq 1$. In particular, we have $\inf\{a_k|k \geq n\} \geq 5 - 1 = 4$. In particular, we have $\inf\{a_k|k \geq N\} \geq 4$. Then, by definition of inf, we have $a_k \geq 4$ for every $k \geq N$.

(d) Clearly, the sequence is bounded above by 100 and below by 0. So, lim sup and lim inf are finite. For $n \geq 50$, we have $\frac{100}{n} \leq 2$. Thus, for every $k \geq 50$, $b_k \leq 2$. So, by how the sequence is defined, for every $n \geq 50$, we have $\sup\{b_k|k \geq n\} \leq 2$. This shows that $\limsup b_n \leq 2$. On the other hand, since $\{k|k \geq n \text{ and } 3|k - 1\} \subset \{k|k \geq n\}$, we have

$$\sup\{b_k|k \geq n\} \geq \sup\{b_k|k \geq n \text{ and } 3|k - 1\} = \sup\{2 - \frac{1}{k}|k \geq n \text{ and } 3|k - 1\} = 2.$$

So, we have $\limsup b_n = 2$. Notice that, for $n \geq 50$, the sequence $\sup\{b_k|k \geq n\}$ the sequence becomes constant with value 2. If we define $n_k = 3k + 1$, we have that (b_{n_k}) converges to 2, as $b_{n_k} = 2 - \frac{1}{3k+1}$.

Now, since $b_n \geq 0$ for every n, we have $\liminf b_n \geq 0$. We claim that the sequence $\inf\{b_k|k \geq n\}$ is constant with value 0. Indeed, we have

$$0 \leq \inf\{b_k|k \geq n\} \leq \sup\{b_k|k \geq n \text{ and } 3|k\} = \inf\{\frac{100}{k}|k \geq n \text{ and } 3|k\} = 0,$$

where the first inequality follows from the fact that (b_n) is non-negative, the second inequality from the fact that $\{k|k \geq n \text{ and } 3|k\} \subset \{k|k \geq n\}$. This shows the claim and that $\liminf b_n = 0$. Then, if we define $m_k = 3_k$, the subsequence (b_{m_k}) converges to 0, as $b_{m_k} = \frac{100}{3k}$.

9. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

(a) If

$$\lim_{n\to\infty} a_n = 0,$$

then

$$\lim_{n\to\infty} (a_n \sin(n)) = 0.$$

(b) If (a_n) is bounded, then

$$\lim_{n\to\infty} (a_n e^{-n}) = 0.$$

(c) If

$$\lim_{n\to\infty} a_n = 0,$$

then the sequence $b_n := a_n e^n$ is unbounded.

Solution:

(a) True. Note $-a_n \le a_n \sin(n) \le a_n$ for all n, as the sin function is bounded in $[-1, 1]$. The result follows from applying the Squeeze Theorem.

(b) True. By the boundedness of the sequence $\exists C > 0$ s.t. $-Ce^{-n} \le a_n e^{-n} \le Ce^{-n}$. The result follows from applying the Squeeze Theorem, as $\lim_{n\to\infty} e^{-n} = 0$.

(c) False. Take $a_n = e^{-n}$. Then $b_n = 1,\ \forall n \in \mathbb{N}$.

10. Compute the following limits:

(a) $\lim_{n\to\infty} \frac{2^n - 3^n}{3^n + 1}$

(b) $\lim_{n\to\infty} n^3 \left(1 - \cos(\frac{1}{n})\right) \sin(\frac{1}{n})$

(*Hint: Use the fact that* $\lim_{m\to\infty} \frac{\sin(\frac{1}{m})}{\frac{1}{m}} = 1$ *and* $\lim_{m\to\infty} \cos\left(\frac{1}{m}\right) = 1$.)

(c) $\lim_{n\to\infty} \frac{\sin^2(n)}{2^n}$

(d) $\lim_{n\to\infty} n(\sqrt{n^4 + 6n + 3} - n^2)$

Solution:

(a)

$$\lim_{n\to\infty} \frac{2^n - 3^n}{3^n + 1} = \lim_{n\to\infty} \frac{3^n((\frac{2}{3})^n - 1)}{3^n(1 + \frac{1}{3^n})} = \lim_{n\to\infty} \frac{(\frac{2}{3})^n - 1}{1 + ((\frac{1}{3})^n}.$$

The geometric sequences $\left(\frac{2}{3}\right)^n$ and $\left(\frac{1}{3}\right)^n$ converge to 0, as their ratio is strictly between 0 and 1. Thus, the numerator converges to -1, and the denominator to 1; since the limit of the denominator is not zero, the ratio converges to the ration of the limits which is -1.

(b) We use the equalities

$$\sin^2\left(\frac{1}{n}\right) = 1 - \cos^2\left(\frac{1}{n}\right) = \left(1 - \cos\left(\frac{1}{n}\right)\right)\left(1 + \cos\left(\frac{1}{n}\right)\right)$$

to get

$$\lim_{n\to\infty} n^3 \left(1 - \cos\left(\frac{1}{n}\right)\right) \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \left(1 + \cos\left(\frac{1}{n}\right)\right)^{-1} \frac{\sin^3\left(\frac{1}{n}\right)}{n^{-3}} = \frac{1}{2},$$

where we use the two facts given in the hint to get that the limit of the first factor is $\frac{1}{2}$ and of the second factor is 1.

(c) We have

$$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$$

and by Squeeze Theorem $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$.

(d)

$$\begin{aligned}\lim_{n\to\infty} n(\sqrt{n^4+6n+3}-n^2) &= \lim_{n\to\infty} n(\sqrt{n^4+6n+3}-n^2)\frac{\sqrt{n^4+6n+3}+n^2}{\sqrt{n^4+6n+3}+n^2} \\ &= \lim_{n\to\infty} \frac{n(n^4+6n+3-n^4)}{\sqrt{n^4+6n+3}+n^2} \\ &= \lim_{n\to\infty} \frac{n(6n+3)}{n^2(\sqrt{1+6/n^3+3/n^4}+1)} = \frac{6}{2} = 3.\end{aligned}$$

11. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

(a) If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$

then (a_n) converges.

(b) If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$

then (a_n) diverges.

Solution:

(a) False, take $a_n = n$.

(b) False, take for example $a_n = 1/n$. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$

but $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

12. Determine if the sequence (a_n) is convergent or not in the following cases.

1. $a_n = \frac{n}{e^n}$.
2. $a_n = \frac{10^n}{n!}$
3. $a_n = \frac{n^n}{e^n}$
4. $a_n = \frac{n!}{n^n e^{\frac{n}{2}}}$

Solution:

1. We compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{e^{n+1}}\frac{e^n}{n} = \frac{n+1}{n}\frac{1}{e}.$$

   Hence, $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{e} < 1$ and the sequence (a_n) is convergent by the quotient criterion.

2. We compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{(n+1)!}\frac{n!}{10^n} = \frac{10}{n+1}.$$

   Hence, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ and the sequence (a_n) is convergent by the quotient criterion.

3. We compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{e^{n+1}}\frac{e^n}{n^n} = \frac{1}{e}\frac{(n+1)^{n+1}}{n^n} = \frac{1}{e}\left(\frac{n+1}{n}\right)^n(n+1).$$

   Hence, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} n+1 = +\infty$ and the sequence (a_n) is unbounded by the quotient criterion.

4. We compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)!}{(n+1)^{n+1}e^{\frac{n+1}{2}}}}{\frac{n!}{n^n e^{\frac{n}{2}}}} = \frac{(n+1)!}{(n+1)^{n+1}e^{\frac{n+1}{2}}}\cdot\frac{n^n e^{\frac{n}{2}}}{n!}$$
$$= \frac{(n+1)!}{n!}\cdot\frac{n^n}{(n+1)^{n+1}}\cdot\frac{e^{\frac{n}{2}}}{e^{\frac{n+1}{2}}} = \frac{n+1}{n+1}\cdot\frac{n^n}{(n+1)^n}\cdot\frac{1}{e^{\frac{1}{2}}} = \frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{e^{1/2}}.$$

   Then, the sequence $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ is convergent because $\left(\left(1+\frac{1}{n}\right)^n\right)$ is a convergent sequence with limit $\neq 0$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{e^{1/2}} = \frac{1}{e^{3/2}} < 1$$

   and the sequence (a_n) is convergent by the quotient criterion.

5.

13. Determine if the following sequences converge or not. If the sequence is convergent, determine its limit.

    (a) $a_n = \frac{3n^2-1}{10n+5n^2}$

    (b) $a_n = \frac{3^{2n}}{n}$

    (c) $a_n = \frac{(-1)^n n^2}{2^n}$

Solution:

(a) Since the numerator and the denominator consist of polynomials of degree 2, then the sequence is convergent. We have

$$\lim_{n\to\infty} \frac{3n^2-1}{10n+5n^2} = \lim_{n\to\infty} \frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)} = \frac{3}{5}.$$

(b) We use the quotient criterion:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{3^{2(n+1)}}{n+1}}{\frac{3^{2n}}{n}} = 3^2 \lim_{n\to\infty} \frac{n}{n+1} = 3^2 > 1.$$

Since $\rho > 1$ the sequence is divergent.

(c) We use the quotient criterion:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{1}{2} \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = \frac{1}{2} < 1.$$

Since $\rho < 1$ the sequence (a_n) is convergent and the limit is 0.

14. Compute the following limits:

(a) $\lim_{n\to\infty} \frac{n^3}{7^n} \cos\left(n^2\right)$

(b) $\lim_{n\to\infty} \frac{\sin(n+1)-\sin(n-1)}{\cos(n+1)+\cos(n-1)}$

(*Hint: Use trigonometric formulas from Exercise Sheet 1*)

(c) $\lim_{n\to\infty} \frac{\sin\left(\sqrt{n^3+n^2+1}\right)}{n^3+n^2+1}$

Solution:

(a) First consider the sequence $b_n = \frac{n^3}{7^n}$. By induction we can show that $7^n \geq n^4$ for all $n \geq 6$ (so, the base case will be $n = 6$). The base case $n = 6$ is satisfied by direct inspection. Now suppose that for some $N \geq 6$ we know that $7^N \geq N^4$. We have

$$(N+1)^4 < 2N^4 < 2 \cdot 7^N < 7^{N+1},$$

as the first inequality is true for all $N \geq 6$; induction proof is complete. Then,

$$-\frac{n^3}{7^n} \leq \frac{n^3}{7^n}\cos(n^2) \leq \frac{n^3}{7^n} \implies -\frac{n^3}{n^4} \leq \frac{n^3}{7^n}\cos(n^2) \leq \frac{n^3}{n^4} \implies -\frac{1}{n} \leq \frac{n^3}{7^n}\cos(n^2) \leq \frac{1}{n}.$$

By the Squeeze Theorem we see that $\lim_{n\to\infty} a_n = 0$.

(b) We use the trigonometric formulas in Exercise Sheet 1.

$$\lim_{n\to\infty} \frac{\sin(n+1)-\sin(n-1)}{\cos(n+1)+\cos(n-1)} = \lim_{n\to\infty} \frac{2\cos(n)\sin(1)}{2\cos(n)\cos(1)} = \tan(1)\ .$$

(c) We have

$$\lim_{n\to\infty} \frac{\sin\left(\sqrt{n^3+n^2+1}\right)}{n^3+n^2+1} = 0 \ ,$$

because $\left|\sin\left(\sqrt{n^3+n^2+1}\right)\right| \le 1$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{1}{n^3+n^2+1} = 0 \ .$$

15. Give an example of a sequence (x_n) such that the sequence $y_n = x_{n+1} - x_n$ converges to 0 but (x_n) itself is divergent.

Solution:

Take the sequence $x_n = \sqrt{n}$, $n \in \mathbb{N}$. x_n is clearly divergent but the sequence $y_n = \sqrt{n+1} - \sqrt{n}$ converges to zero:

$$\lim_{n\to+\infty} y_n = \lim_{n\to\infty} y_n \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = 0.$$

16. Prove that if $\lim_{n\to\infty} x_n = +\infty$ and (y_n) bounded from below, then $\lim_{n\to\infty}(x_n + y_n) = +\infty$. Show that this is true also if $+\infty$ is replaced with $-\infty$ and (y_n) is assumed to be bounded from above.

Solution:

Let C be a lower bound for the values of (y_n), that is, $C \le y_n$, $\forall n \in \mathbb{N}$. Now, we need to show that, for every $M > 0$, there exists $N \in \mathbb{N}$ such that, if $n \ge N$, then $x_n + y_n \ge M$. Fix $M > 0$, and consider $M + |C| > 0$. As $\lim_{n\to\infty} x_n = +\infty$, there exists N such that, if $n \ge N$, then $x_n \ge M + |C|$. Then, for every $n \ge N$, we have $x_n + y_n \ge M + |C| + C \ge M$. Thus, $x_n + y_n \to +\infty$ as $n \to \infty$ as required.

Now, we address the second part of the question. Let C be an upper bound for the values of (y_n), that is, $C \ge y_n \quad \forall n \in \mathbb{N}$. Then $x_n + y_n \le x_n + C \to -\infty$ as $n \to \infty$ and thus $x_n + y_n \to -\infty$ as $n \to \infty$ as required.

17. Prove that if $x_n \neq 0$, for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} |\frac{x_n}{x_{n-1}}| = +\infty$ then (x_n) is unbounded and, thus it diverges. Construct examples of sequences (x_n) satisfying the conditions above and such that $\lim_{n\to\infty} x_n = +\infty$ (resp. $\lim_{n\to\infty} x_n = -\infty$).

Solution: By definition of divergence to $+\infty$, we have the following: for every $M > 0$, there exists $N \in \mathbb{N}$ such that, if $n \ge N$, then $\left|\frac{x_n}{x_{n-1}}\right| \ge C$. Now, let $C = 2$ and fix N accordingly. Then, for every $n \ge N$, by clearing denominators, we have $|x_n| \ge 2|x_{n-1}|$.

By induction, we show the following: for every $n \ge N$, $|x_n| \ge 2^{n-N}|x_N|$. We check the base case $n = N$: $|x_N| \ge |x_N| = 2^0|x_N| = 2^{N-N}|x_N|$. Now, fix $n \ge N$, and assume $|x_n| \ge 2^{n-N}|x_N|$. By what we showed in the previous paragraph, as $n+1 \ge N$, we have $|x_{n+1}| \geq 2|x_n|$. Applying the inductive hypothesis to $|x_n|$, we get $|x_{n+1}| \geq 2|x_n| \geq 2 \cdot 2^{n-N}|x_N| = 2^{n+1-N}|x_N|$. So, the inductive step is settled.

Since $|x_N| \neq 0$ and $\lim\limits_{m\to\infty} 2^m = +\infty$, we get that $\lim\limits_{n\to\infty} 2^{n-N}|x_N| = +\infty$. Since $|x_n| \geq 2^{n-N}|x_N|$ for every $n \geq N$, it follows that $\lim\limits_{n\to\infty} |x_n| = +\infty$. In particular, (x_n) is not bounded.

Examples are $x_n = e^{n^n}$ or $x_n = -e^{n^n}$ or $x_n = n!$ or $x_n = -n!$ among others.

18. Consider the recursive sequence $a_{n+1} = 7 - \frac{10}{a_n}$, with initial datum $a_1 = 4$. Compute the first three values. Then, show that it is bounded by 2 and 5, that it is increasing, and then compute the limit.

Solution:

The first values are 4, 4.5 and 4.778. We prove that it is bounded by 2 and 5 using induction. The claim is true for a_1. Now we have tho show that if a_n is bounded by 2 and 5, the same is true for a_{n+1}; this is done by explicit computation as follows:

$$a_{n+1} = 7 - \frac{10}{a_n} < 7 - \frac{10}{5} = 5, \quad a_{n+1} = 7 - \frac{10}{a_n} > 7 - \frac{10}{2} = 2.$$

To show that the sequence is increasing, we have to show that $a_n - a_{n+1}$ is non-positive for all n. This difference is equal to $a_n - 7 + \frac{10}{a_n}$, and it is negative for all a_n between 2 and 5 (solve the inequality $x - 7 + \frac{10}{x} < 0$). Since a_n is always between 2 and 5, we get the claim.

By monotone convergence we now know that the limit exists. Let us call it L. Taking the limit on both side of the equality $a_{n+1} = 7 - \frac{10}{a_n}$, we get that

$$L = 7 - \frac{10}{L}$$

So L is equal either to 2 or to 5. But now we recall that the sequence is increasing, and starts at 4, so $L = 5$.

19. Consider the recursive sequence $a_{n+1} = \sqrt{8a_n - 7}$, with initial datum $a_1 = 4$. Show that it is bounded by 1 and 7, that it is increasing, and then compute the limit.

Solution:

Same strategy as in Exercise 18. We prove by induction that the sequence is bounded by 1 and 7. We observe that $a_1 = 4$ is between 1 and 7. We assume that $1 \leq a_n \leq 7$ and we prove the same inequalities for a_{n+1} as follows:

$$a_{n+1} = \sqrt{8a_n - 7} \leq \sqrt{8 \cdot 7 - 7} = 7, \quad a_{n+1} = \sqrt{8a_n - 7} \geq \sqrt{8 \cdot 1 - 7} = 1.$$

The sequence is increasing if $a_{n+1} - a_n$ is non-negative. We observe that

$$a_{n+1} - a_n = \frac{a_{n+1}^2 - a_n^2}{a_{n+1} + a_n} = \frac{8a_n - 7 - a_n^2}{a_{n+1} + a_n}$$

is positive for a_n between 1 and 7. Hence, the sequence is increasing.

We know that the limit exists by monotone convergence. Let $L = \lim_{n\to\infty} a_n$. Then by taking the limit of both sides of the equation $a_{n+1}^2 = 8a_n - 7$ we get $L^2 = 8L - 7$. Hence, $L \in \{1, 7\}$. Since a_n is increasing and $a_1 = 4$, we conclude that $L = 7$.

20. Find the limit for $x_n = \frac{\sin(x_{n-1})}{2}, x_0 = 1$. [*Hint:* use the fact $|\sin(x)| \le |x|$ for all x]

Solution:
Note $|x_n| = |\frac{\sin(x_{n-1})}{2}| \le |\frac{x_{n-1}}{2}|$. Thus, by induction, we have that $|x_n| \le \frac{|x_0|}{2^n} = \frac{1}{2^n}$ for all n. Since $\lim_{n\to\infty} \frac{1}{2^n} = 0$, by the squeeze theorem, we conclude $\lim_{n\to 0} x_n = 0$.

21. Let $(a_n), (b_n)$ be sequences. State if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

(a) If (a_n) is monotone, then $\lim_{n\to\infty} a_n$ exists or $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$.

(b) If (a_n) and (b_n) are monotone, then the sequence $c_n = a_n + b_n$ is monotone.

(c) If $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$, then (a_n) is a bounded sequence.

(d) An unbounded sequence can have a convergent subsequence.

(e) If (a_n) has no convergent subsequence, then (a_n) is unbounded.

Solution:

(a) True. Indeed, if (a_n) is monotone and bounded, then it converges and hence $\lim_{n\to\infty} a_n$ exists. If it is monotone increasing and unbounded, then it approaches $+\infty$ and so $\lim_{n\to\infty} a_n = +\infty$. If it is monotone decreasing and unbounded, then it approaches $-\infty$ and so $\lim_{n\to\infty} a_n = -\infty$.

(b) False. For example, take $a_n = 2n + (-1)^n$ and $b_n = -2n$. Then (a_n) is monotone increasing because

$$a_{n+1} - a_n = 2(n+1) + (-1)^{n+1} - 2n - (-1)^n = 2 + 2(-1)^{n+1} \ge 2 - 2 = 0,$$

and (b_n) is monotone decreasing. But $c_n = a_n + b_n = (-1)^n$ is not monotone.

(c) False. Take for example $a_n = \sqrt{n}$ for all $n \in \mathbb{N}$. Then

$$|a_{n+1} - a_n| = \sqrt{n+1} - \sqrt{n} = \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

converges to 0 but (a_n) is not bounded.

(d) True. Take for example

$$a_n = \begin{cases} n^2 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

then (a_{2k}) is a convergent subsequence, while a_n is divergent.

(e) True. This statement is the contrapositive of the Bolzano–Weierstrass theorem.