Analysis 1 - Exercise Set 4

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Compute

   (a) $(1+i\sqrt{3})^{1980}$

   (b) $(1+i\sqrt{3})^{1988}$

Solution:

(a) We have
$$(1+\sqrt{3}i)^{1980} = 2^{1980}e^{i\frac{1980\pi}{3}} = 2^{1980}$$

(b) We have
$$(1+\sqrt{3}i)^{1988} = 2^{1988}e^{i\frac{1988\pi}{3}} = 2^{1988}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) =$$
$$= 2^{1988}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2^{1987}(-1+\sqrt{3}i).$$

2. Find all the solution of the following equations in $\mathbb{C}$. [The unknown is $z = x+iy$, or, if you prefer you could use polar form.]

   (a) $z^2 = i$

   (b) $z^5 = 1$.

   (c) $z^2 = -3+4i$.

Solution:

(a) We can either use Euler's formula or we can solve it directly as described here. We are searching for $x+iy$ such that $(x+iy)^2 = i$ meaning $(x^2-y^2)+i(2xy) = i$. Clearly $x^2-y^2 = 0$ and $2xy = 1$. From the first equation we deduce that $x = \pm y$.
$$x = y \Longrightarrow x\cdot(x) = \frac{1}{2} \Longrightarrow x = \pm\frac{\sqrt{2}}{2}$$
$$x = -y \Longrightarrow x\cdot(-x) = \frac{1}{2} \Longrightarrow x^2 = -\frac{1}{2} \quad \text{(not valid since } x\in\mathbb{R}).$$
So the roots of i are $\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$.

To check the solution, one can compute $\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^2 - i$ and $\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)^2 - i$.

(b) Write z using Euler's formula: $z = |z|e^{i\theta}$ where θ is the phase of z. Hence, $z^5 = |z|^5 e^{i5\theta} = e^{i2\pi k} \Rightarrow |z| = 1$ and $\theta = \frac{2\pi}{5}k$ for $k = 0, 1, 2, 3, 4$.

(c) Write z using Euler's formula: $z = |z|e^{i\theta}$. Note $|-3+4i| = 5$ thus $|z|^2 = 5 \Rightarrow |z| = \sqrt{5}$. Note $2\theta = \arg(-3+4i) = \arctan(-\frac{4}{3}) + \pi + 2\pi k = \varphi + 2\pi k$. Thus, $\theta = \frac{\varphi}{2} + \pi k$ for $k = 0, 1$.

3. In the context of complex numbers, state if the following statements are true or false.

   (a) There exists a $n \in \mathbb{N}$ such that $(1 + i\sqrt{3})^n$ is pure imaginary.

   (b) There exists a positive $n \in \mathbb{N}$ such that $(1 - i\sqrt{3})^n$ is real.

Solution:

(a) False. We have

$$(1 + \sqrt{3}i)^n = 2^n e^{i\frac{n\pi}{3}} = 2^n \left(\cos(n\frac{\pi}{3}) + i\sin(n\frac{\pi}{3})\right)$$

For the complex number to be pure imaginary, we require $\cos(n\frac{\pi}{3}) = 0$ which means $n = \frac{3}{2} + 3k$ for some $k \in \mathbb{Z}$. This condition cannot be satisfied if $n \in \mathbb{N}^*$.

(b) True. Similar to the previous part we have

$$(1 - \sqrt{3}i)^n = 2^n e^{-i\frac{n\pi}{3}} = 2^n \left(\cos(n\frac{\pi}{3}) - i\sin(n\frac{\pi}{3})\right)$$

it is sufficient to find some n such that $\sin(n\frac{\pi}{3}) = 0$. Take for example $n = 3$.

4. Show that for all $\theta \in \mathbb{R}$ and for all $n \in \mathbb{N}$

$$(\cos(\theta) + i\sin(\theta))^n = (\cos(n\theta) + i\sin(n\theta)).$$

Solution: By definition of polar representation, we have $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Then, by the property of the exponents, we have

$$(\cos(\theta) + i\sin(\theta))^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

5. Prove that for all $z_1, z_2 \in \mathbb{C}$,

   (a) $z_1 = 0$ if and only if $|z_1| = 0$.

   (b) $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{i(\alpha_1 - \alpha_2)}$, where $z_2 \neq 0$ and

$$z_1 = |z_1|e^{i\alpha_1}, \quad z_2 = |z_2|e^{i\alpha_2}, \quad \alpha_1,\ \alpha_2 \in \mathbb{R}$$

   are the polar forms of the z_i using Euler's formula.

   (c) $|\frac{z_1}{|z_1|}| = 1$.

Solution:

(a) One direction is trivial. The other follows from if $z = x + iy$, then $0 = |z| = \sqrt{x^2 + y^2} \geq |x| \Rightarrow x = 0$ and similar for y.

(b) $\frac{z_1}{z_2} = \frac{|z_1|e^{i\alpha_1}}{|z_2|e^{i\alpha_2}} = \frac{|z_1|}{|z_2|}e^{i(\alpha_1 - \alpha_2)}$.

(c) Say $z_1 = a + bi \implies |z_1| = \sqrt{a^2 + b^2} \implies \frac{z_1}{|z_1|} = \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}i$ $\implies |\frac{z_1}{|z_1|}| = \sqrt{(\frac{a}{\sqrt{a^2+b^2}})^2 + (\frac{b}{\sqrt{a^2+b^2}})^2} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = 1$.

6. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $\text{Im}(z(2 - i)) = 1$ is

(a) A point.
(b) A line.
(c) A circle.
(d) Empty.

Solution: (b) is correct. Writing $z = x + iy$, we get that $(x+iy)(2-i) = (2x+y) + i(2y-x)$ so the equation becomes

$$2y - x = 1$$

whose solutions are the points on the line.

7. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $\bar{z} = i(z - 1)$ is

(a) A point.
(b) A line.
(c) A circle.
(d) Empty.

Solution: (d) is correct. Write $z = x + iy$, with $x, y \in \mathbb{R}$ then the equation becomes,

$$x - iy = i(x + iy - 1) \implies x - iy = ix - y - i$$

Meaning $x = -y$ and $-y = x - 1$ which has no solutions; we conlude that the equation has no solution.

8. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $z^2 \cdot \bar{z} = z$ is

(a) A point.
(b) A circle.
(c) A point and a circle.
(d) A disk.

Solution: (c) is correct. We can write the equation as $z \cdot ((z\bar{z}) - 1) = 0$. This means that one solution is $z = 0$ which is one point. Also since $z\bar{z} = |z|^2$ another set of solution is $|z|^2 = 1$ which are all points of the circle of radius 1 centered at the origin.

9. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $|z + 3i| = 3|z|$ is
   (a) A point.
   (b) A line.
   (c) A circle.
   (d) Empty.

Solution:
(c) is correct. We square both terms and write $z = x + iy$ and we obtain

$$|z + 3i|^2 = |x + i(y + 3)|^2 = x^2 + (y + 3)^2, \qquad (3|z|)^2 = 9(x^2 + y^2)$$

So the equation turns into

$$x^2 + (y + 3)^2 = 9(x^2 + y^2) \iff x^2 + y^2 - \frac{3}{4}y = \frac{9}{8} \iff x^2 + \left(y - \frac{3}{8}\right)^2 = \left(\frac{9}{8}\right)^2.$$

Then the solution are all the points of the circle of radius 9/8 centered at (0, 3/8).

10. Given the function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = \frac{1+iz}{iz+i}$
   (a) find the domain of the function f. That is, determine the set $\mathrm{Dom}(f) \subseteq \mathbb{C}$ such that $z \in \mathrm{Dom}(f)$ if and only if $f(z)$ is defined;
   (b) find all complex numbers z such that $f(z) = z$;
   (c) find the preimages of $3 + i$.

Solution:

(a) Since f is a rational function (i.e., ratio of two polynomials), we need to determine when the denominator is not 0. So, $z \in \mathrm{Dom}(f)$ if and only if $iz + i \neq 0$. Dividing by i, we get $z + 1 \neq 0$. So, we conclude that $\mathrm{Dom}(f) = \mathbb{C} \setminus \{-1\}$.

(b) You have to solve the equation $f(z) = z$, so

$$\frac{1 + iz}{iz + i} = z$$

which turns out to be

$$z^2 = -i, \quad z \neq -1$$

the two solutions are $z = \pm\frac{1}{\sqrt{2}}(1 - i)$. Verification: substitute the solutions into the equation and compute. For example:

$$\frac{1 + i\frac{1-i}{\sqrt{2}}}{i\frac{1-i}{\sqrt{2}} + i} - \frac{1 - i}{\sqrt{2}} = \frac{\sqrt{2} + i + 1}{i + 1 + \sqrt{2}i} - \frac{1 - i}{\sqrt{2}} = \frac{2 + \sqrt{2}i + \sqrt{2} - i - 1 - \sqrt{2}i - 1 + i + -\sqrt{2}}{\sqrt{2}(i + 1 + \sqrt{2}i)} = 0.$$

(c) You have to solve

$$\frac{1+iz}{iz+i} = 3+i,$$

which is equivalent to

$$1 + iz = (3i-1)z + 3i - 1, \quad z \neq -1.$$

This is linear in z, and the unique solution is $-\frac{8}{5} - \frac{1}{5}i$. Verification: compute $f(-\frac{8}{5} - \frac{1}{5}i)$.

11. (a) Prove that for all $n \in \mathbb{N}$, $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$.
    (b) Find all the solutions to the equation $x^n = 1$, for $n \in \mathbb{N}$.

Solution:

(a) We use the exponential representation of complex numbers:

$$(\cos(x) + i\sin(x))^n = (e^{ix})^n = e^{inx} = \cos(nx) + i\sin(nx).$$

(b) We know that raising a complex number to a power also raises its modulus to the same power. So, to be a solution to $x^n = 1$, we need $|x| = 1$. So, we have $x = e^{i\theta}$ for some angle θ. Then, by part (a), we get that $\cos(n\theta) + \sin(n\theta) = 1$. So, we have $n\theta = 2k\pi$ for some $k \in \mathbb{Z}$. Thus, the solutions are $\theta = 0, \frac{2\pi}{n}, 2\frac{2\pi}{n}, \ldots, (n-1)\frac{2\pi}{n}$, and the solutions for x are $x = 1, x^{\frac{2\pi}{n}}, x^{2\frac{2\pi}{n}}, \ldots, x^{(n-1)\frac{2\pi}{n}}$.

12. Find all the solutions of the following equations in $\mathbb{C}$.
    (a) $z^2 + 6z + 12 - 4i = 0$
    (b) $(z^3 - 1)^2 = -1$

Solution:

(a) First, we try completing the square (using the quadratic formula is analogous). So, we have

$$z^2 + 6z + 12 - 4i = z^2 + 6z + 9 + 3 - 4i = (z+3)^2 + 3 - 4i.$$

So, our equation becomes

$$(z+3)^2 = 4i - 3.$$

With some work involving trigonometry, one can show that $1 + 2i$ is a square root of $4i - 3$. Then, the other root is $-1 - 2i$. So, one solution will be $z + 3 = 1 + 2i$, namely $z = -2 + 2i$, and the other will be $z = -4 - 2i$. In this process, we relied on the given fact that $1 + 2i$ is a square root of $4i - 3$. This may require some lengthy work. For this reason, sometimes it is best to consider an alternative approach, which is discussed here below.

We consider the form $z = a + ib$ with $a, b \in \mathbb{R}$. Substituting this into the equation we get,

$$(a + ib)^2 + 6(a + ib) + 12 - 4i = 0.$$

which is equivalent to the system of equations

$$a^2 - b^2 + 6a + 12 = 0$$
$$2ab + 6b - 4 = 0.$$

from the first equation we obtain

$$a = -3 \pm \sqrt{b^2 - 3}$$

and since a and b are real numbers then $|b| \geq \sqrt{3}$. We can write the second equation as

$$a = \frac{2}{b} - 3$$

If we substitute this in the above expression we get that

$$b^2 - 3 = \frac{4}{b^2} \implies b^4 - 3b^2 - 4 = (b^2 - 4)(b^2 + 1) = 0$$

Then we have $b = \pm 2$ since b is real. So the two solutions to the equations are $z_1 = -2 + 2i$ and $z_2 = -4 - 2i$.

Verification: substitute the solutions into the equation and compute.

(b) Use the change of variable $\tilde{z} = z^3 - 1$. Now the equation for $\tilde{z}$ takes the form

$$\tilde{z}^2 = -1$$

so $\tilde{z} = \pm i$. So $z^3 = 1 \pm i$. We write

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$
$$1 - i = \sqrt{2}e^{-i\frac{\pi}{4}}$$

We use the polar representation of complex numbers and write $z = re^{i\theta}$ with $r > 0$ and $0 \leq \theta < 2\pi$. For $z^3 = \sqrt{2}e^{i\pi/4}$ we have

$$z^3 = (re^{i\theta})^3 = r^3 e^{i(3\theta)} = \sqrt{2}e^{i(2k\pi + \pi/4)}, \quad k = 0, 1, 2, \ldots$$

which has the solutions $r = \sqrt[6]{2}$ and $\theta = \frac{\pi}{12}$, $\theta = \frac{3\pi}{4}$ and $\theta = \frac{17\pi}{12}$. Similarly for $z^3 = \sqrt{2}e^{-i\pi/4}$ we have

$$z^3 = (re^{i\theta})^3 = r^3 e^{i(3\theta)} = \sqrt{2}e^{i(2k\pi - \pi/4)}, \quad k = 0, 1, 2, \ldots$$

which has the solutions $r = \sqrt[6]{2}$ and $\theta = \frac{23\pi}{12}$, $\theta = \frac{7\pi}{12}$ and $\theta = \frac{5\pi}{4}$. The final 6 solutions are

$$z_1 = \sqrt[6]{2}e^{i\frac{\pi}{12}}$$
$$z_2 = \sqrt[6]{2}e^{i\frac{3\pi}{4}}$$
$$z_3 = \sqrt[6]{2}e^{i\frac{17\pi}{12}}$$
$$z_4 = \sqrt[6]{2}e^{i\frac{23\pi}{12}}$$
$$z_5 = \sqrt[6]{2}e^{i\frac{7\pi}{12}}$$
$$z_6 = \sqrt[6]{2}e^{i\frac{5\pi}{4}}$$

Verification: substitute the solutions into the equation and compute.

13. (a) Let $\{a_n\}$ be a sequence and let $\{b_n\}$ be the sequence defined as $b_n := |a_n|$. Prove that $\{a_n\}$ is bounded if and only if so is $\{b_n\}$.

(b) Prove that if $q < -1$, then the sequence $x_n = aq^n$, $a \in \mathbb{R}$, $a \neq 0$, is unbounded.

(c) Provide an example of a sequence $\{a_n\}$ that is bounded above and such that $\{|a_n|\}$ is not bounded above.

Solution:

1. Let us prove that the boundedness of $\{a_n\}$ implies the boundedness of $\{b_n\}$. The sequence $\{a_n\}$ is bounded $\Longleftrightarrow_{\text{by definition}}$ there exists real number $C_1, C_2 \in \mathbb{R}$ such that
$$C_1 \leq a_n \leq C_2, \quad \forall n \in \mathbb{N}.$$
Let $C := \max\{|C_1|, |C_2|\}$. Then,
$$-C \leq C_1 \leq a_n \leq C_2 \leq C, \quad \forall n \in \mathbb{N}, \tag{1}$$
by the properties of the aboslute values of real numbers. We can rewrite (1) as
$$|a_n| \leq C, \quad \forall n \in \mathbb{N}. \tag{2}$$
As $|a_n| \geq 0$, then $0 \leq |a_n| \leq C$, for all $n \in \mathbb{N}$ and b_n is bounded.
Let us prove that the boundedness of $\{b_n\}$ implies the boundedness of $\{a_n\}$. The sequence $\{b_n\}$ is bounded $\Longleftrightarrow_{\text{by definition}}$ there exists a non-negative real number C' such that $0 \leq b_n \leq C'$, $\forall n \in \mathbb{N}$ (since $b_n \geq 0$) $\Longleftrightarrow_{b_n=|a_n|}$ there exists a non-negative real number C' such that $-C' \leq a_n \leq C'$ $\forall n \in \mathbb{N}$ $\Longleftrightarrow_{\text{by definition}}$ (a_n) is bounded.

2. We consider the sequence $|x_n| = a|q|^n$, where now we know $|q| > 1$. Then, we saw in class that this sequence is not bounded (using Bernoulli's inequality). Then, by part (a) we are done.

3. The sequence $a_n = -n$ is a valid example.

14. Let $\{x_n\}$ be the recursive sequence defined as
$$\begin{cases} x_n = x_{n-1} + (-1)^n n^2 \\ x_0 = 0. \end{cases}$$
Prove that for all $m \in \mathbb{N}$
$$\begin{cases} x_{2m} = (2m+1)m \\ x_{2m+1} = -(2m+1)(m+1). \end{cases}$$
[*Hint*: use induction on m].
Is $\{x_n\}$ bounded from above? Is it bounded from below?

Solution:

We will prove this by induction. One can easily verify the formula for $m = 0$: indeed, $0 = x_0 = x_{2\cdot 0} = (2 \cdot 0 + 1) \cdot 0$.

Now suppose the claim holds for some natural number $m = k$; that is, we have $x_{2k} =$

$(2k+1)k$ and $x_{2k+1} = -(2k+1)(k+1)$. Then,

$$\begin{aligned} x_{2k+2} =& x_{2k+1} + (-1)^{2k+2}(2k+2)^2 \\ =& -(2k+1)(k+1) + (2k+2)^2 \\ =& (2k+3)(k+1) \end{aligned}$$

and

$$\begin{aligned} x_{2k+3} =& x_{2k+2} + (-1)^{2k+3}(2k+3)^2 \\ =& (2k+3)(k+1) - (2k+3)^2 \\ =& -(2m+3)(m+2) \end{aligned}$$

as required. From this we also see that the sequence is not bounded from below nor from above, since x_{2m} is monotonic increasing and unbounded and x_{2m+1} is monotonic decreasing and unbounded.