Analysis 1 - Exercise Set 3

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Let $[x]$ denote the integral part of a number $x \in \mathbb{R}$.
   Prove that, for every $x \in \mathbb{R}$, $[x] = -[-x]$.

Solution: Recall the definition of integral part:

$$[x] = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{if } x \leq 0. \end{cases}$$

Thus, regardless of the sign of x, we have $x = [x] + a$, where $0 \leq a < 1$.

First, assume $x \geq 0$. Then, $x = [x] + a = \lfloor x \rfloor + a$ with $0 \leq a < 1$. Then, we have $-x = -\lfloor x \rfloor - a$. Since $\lfloor x \rfloor$ is an integer, then so is $-\lfloor x \rfloor$. Then, as $0 \leq a < 1$, we have $-1 < -a \leq 0$. Thus, $-\lfloor x \rfloor$ is the smallest integer that is greater than or equal to $-x$. That is, $-\lfloor x \rfloor = \lceil -x \rceil = [-x]$, where the first equality comes from the definition of ceiling as smallest integer greater than or equal to the input, and the second equality follows from the definition of integral part of a non-positive number. Since $\lfloor x \rfloor = [x]$ as $x \geq 0$, we get

$$[x] = \lfloor x \rfloor = -\lceil -x \rceil = -[-x].$$

This proves the claim for $x \geq 0$.

Now, assume $x \leq 0$. So, we may write $x = -y$, where $y \geq 0$. By the first part, we have $[y] = -[-y]$. As $x = -y$, this gives us $[-x] = -[x]$. Multiplying by -1 both sides gives the sought identity.

2. Let $a, b \in \mathbb{R}$. Prove that $||a| - |b|| \leq |a - b|$ and $||a| - |b|| \leq |a + b|$

Solution: First, by the triangle inequality, we have

$$|a| = |a - b + b| \leq |a - b| + |b|. \tag{1}$$

Similarly, we have

$$|b| = |b - a + a| \leq |a - b| + |a|, \tag{2}$$

where we also used $|a - b| = |b - a|$. Rearranging (1), we get

$$|a| - |b| \leq |a - b|, \tag{3}$$

while rearranging (2), we get

$$|b| - |a| \leq |a - b|. \tag{4}$$

By definition of absolute value, one among $||a| - |b|| = |a| - |b|$ and $||a| - |b|| = |b| - |a|$ holds true. Thus, by (3) and (4), we get $||a| - |b|| \leq |a - b|$.

For the second part, we use the fact that $|-b| = |b|$ and we apply the first part to the real numbers a and $-b$. Indeed, we get

$$||a| - |b|| = ||a| - |-b|| \leq |a - (-b)| = |a + b|.$$

3. Compute $\sup S$ and $\inf S$ where $S \subseteq \mathbb{R}$ is defined as

   (a) $S := \bigcup_{n=1}^{\infty} [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$. Does S admit maximum and/or minimum?

   (b) $S := \bigcap_{n=1}^{\infty} (-1 - \frac{1}{n}, 1 + \frac{1}{n})$. Does S admit maximum and/or minimum?

Solution:

(a) Note $S = (-1, 1)$. Thus, $\inf S = -1$ and $\sup S = 1$. They are not maxima or minima.

(b) Note $S = [-1, 1]$. Thus, $\inf S = -1$ and $\sup S = 1$. They are maxima or minima.

4. Compute $\min S$ where $S \subseteq \mathbb{N}$ is defined as

   (a) $S := \{n \in \mathbb{N} : \sqrt{n} > 17\}$

   (b) $S := \{n \in \mathbb{N} : \sum_{i=1}^{n} i \geq 17\}$

   (c) $S := \{n \in \mathbb{N} : \sum_{i=1}^{n} 2^{-i} > 1.7\}$

Solution:

(a) $n \in S \Leftrightarrow n > 17^2$. Thus, the minimum of S is $17^2 + 1$.

(b) Note $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. First find $x \in \mathbb{R}_+$ s.t. $\frac{x(x+1)}{2} = 17$ which gives $x = \frac{-1+\sqrt{137}}{2}$ from the quadratic formula. Now note $11^2 < 137 < 12^2$ hence $x \in [\frac{-1+11}{2}, \frac{-1+12}{2}] = [5, 5.5]$. Thus, the minimum of S is $\lceil x \rceil = 6$.

(c) Note $\sum_{i=1}^{n} 2^{-i} = \frac{1}{2} \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1 - (\frac{1}{2})^n < 1.7$. Thus, the set is empty and the minimum is not defined.

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5. Compute $\max S$ where $S \subseteq \mathbb{Z}$ is defined as

   (a) $S = \{n \in \mathbb{Z} \mid n \neq 0 \text{ and } n + \frac{20}{n} < 9\}$

   (b) $S = \{n \in \mathbb{Z} \mid (\sqrt{3})^n \leq 10^{17}\}$.

   (c) $S = \{n \in \mathbb{Z} \mid \alpha^n \leq C\}$ where $\alpha > 1$ and $C > 1$ are constants. [You must discuss how $\max S$ varies, when α and C vary.]

Solution:

(a) If $n > 0$, $n + \frac{20}{n} < 9 \Leftrightarrow 0 > n^2 - 9n + 20 = (n-4)(n-5)$ from which we see that if we are restricted to natural numbers this can never be satisfied, as the solution of $0 > x^2 - 9x + 20 = (x-4)(x-5)$ is $4 < x < 5$. So, we may assume $n < 0$. Then, we see that $n + \frac{20}{n}$ is always negative, as $n < 0$. So, the inequality is always satisfied if if $n < 0$. Thus, S coincides with the set of negative integers. So, its maximum is -1.

(b) See the solution for (c) but substitute $C = 10^{17}$ and $\alpha = \sqrt{3}$.

(c) We seek the largest $n \in \mathbb{Z}$ s.t. $\alpha^n \leq C$. The natural logarithm is an increasing function. Thus, taking logarithms both sides of the inequality preserves the inequality, and we get $n \ln(\alpha) = \ln(\alpha^n) \leq \ln(C)$. Thus, the maximum of S is $\lfloor \frac{\ln(C)}{\ln(\alpha)} \rfloor$.

6. For the following complex numbers z compute the real and imaginary part, the complex conjugate $\bar{z}$, the absolute value $|z|$, the argument (also called phase) $\arg(z)$ and the inverse z^{-1}:

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2} i; \quad z = 16i; \quad z = 2 + 3i - 3e^{i\frac{\pi}{2}}; \quad z = e^{-5\pi i} + i.$$

Solution:

(a) $\text{Re}z = \frac{1}{2}$; $\text{Im}z = \frac{\sqrt{3}}{2}$; $\bar{z} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$; $\arg z = \frac{\pi}{3}$; $|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$;

(b) $\text{Re}z = 0$; $\text{Im}z = 16$; $\bar{z} = -16i$; $\arg z = \frac{\pi}{2}$; $|z| = 16$;

(c) $e^{i\frac{\pi}{2}} = i$, then $z = 2$; $\text{Re}z = 2$; $\text{Im}z = 0$; $\bar{z} = 2$; $\arg z = 0$; $|z| = 2$;

(d) $e^{-5\pi i} = -1$, since fot any $n \in \mathbb{Z}$, $e^{(t+2\pi)i} = e^{ti}$ and $e^{-\pi i} = -1$; then $z = -1 + i$; $\text{Re}z = -1$; $\text{Im}z = 1$; $\bar{z} = -1 - i$; $\arg z = \frac{3\pi}{4}$; $|z| = \sqrt{2}$.

7. Write the following complex numbers in the form $x + iy$.

(a) i^{17}

(b) $\frac{4-i}{3-2i}$

(c) $2i(i-1) + \left(\sqrt{3} + i\right)^3 + (1+i)\overline{(1+i)}$

Solution:

(a) $i^{17} = i \cdot i^{16} = i \cdot (i^4)^4 = i \cdot (1)^4 = i$

(b)

$$\frac{4-i}{3-2i} = \frac{4-i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{12+2-3i+8i}{9+4} = \frac{14+5i}{13} = \frac{14}{13} + i\frac{5}{13}$$

(c)

$$2i(i-1) = 2(-1-i) = -2-2i,$$

$$\left(\overline{\sqrt{3}+i}\right)^3 = \left(\sqrt{3}-i\right)^3 = \left(\sqrt{3}-i\right)^2\left(\sqrt{3}-i\right) = \left(3-1-2i\sqrt{3}\right)\left(\sqrt{3}-i\right)$$

$$= \left(2-2i\sqrt{3}\right)\left(\sqrt{3}-i\right) = 2\sqrt{3}-2i-6i-2\sqrt{3} = -8i,$$

$$(1+i)\overline{(1+i)} = |1+i|^2 = 2.$$

So $2i(i-1)+\left(\overline{\sqrt{3}+i}\right)^3+(1+i)\overline{(1+i)} = -2-2i-8i+2 = -10i$.