Analysis 1 - Exercise Set 10

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Let I be an interval, $f\colon I \to \mathbb{R}$ be a continuous function and $f(I)$ the image of I by f. Say if the following statement are true or false.

   (a) If I is bounded, then $f(I)$ is bounded.

   (b) If $I = [a, \infty[$ with $a \in \mathbb{R}$, then f attains its maximum and minimum in I.

   (c) If f is strictly increasing and I is open, then $f(I)$ is open.

Solution:

(a) False. Take for example the function $f\colon]0,1[\to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Then I is bounded but $f(I) =]1, \infty[$ is not bounded.

(b) False. Take for example the function defined by $f(x) = (x-a)\sin(x-a)$. It neither have a minimum nor a maximum on I because for all $n \in \mathbb{N}$, we have $f\big(a + \frac{\pi}{2} + 2\pi n\big) = \frac{\pi}{2} + 2\pi n > n$ and $f\big(a - \frac{\pi}{2} + 2\pi n\big) = \frac{\pi}{2} - 2\pi n < -n$.

(c) True. Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$ such that $I =]a, b[$. Since the function is strictly increasing, we have

$$A := \lim_{x \to a^+} f(x) = \inf\{f(x) : x \in]a,b[\} < \sup\{f(x) : x \in]a,b[\} = \lim_{x \to b^-} f(x) =: B$$

and A, B defined above belong to $\mathbb{R} \cup \{\pm\infty\}$. In particular, $f(I) \subseteq [A, B]$. We observe that if $A \in f(I)$, then there exists $x \in]a, b[$ such that $f(x) = A$. Since f is strictly increasing, then $x \leq a$, which is impossible because $x > a$. Hence $A \notin f(I)$. Similarly we prove that $B \notin f(I)$. So we conclude that $f(I) \subseteq]A, B[$. Now we observe that by definition of limits we can find sequences (x_n), (y_n) contained in $]a, b[$ such that (x_n) converges to a and (y_n) converges to b and $x_n \leq y_n$ for all $n \in \mathbb{N}$, so that $(f(x_n))$ converges to A and $(f(y_n))$ converges to B, and hence $]A, B[= \cup_{n \in \mathbb{N}} [f(x_n), f(y_n)]$. By the intermediate value theorem, we have $[f(x_n), f(y_n)] \subseteq f(I)$ for all $n \in \mathbb{N}$. Then

$$]A, B[= \cup_{n \in \mathbb{N}} [f(x_n), f(y_n)] \subseteq f(I).$$

We proved that $f(I)$ is the open interval $]A, B[$.

2. Find, if it exists, continuous extension of the function $f :]2, \infty[\to \mathbb{R}$ give by $f(x) = \frac{\sqrt{x} - \sqrt{2} + \sqrt{x-2}}{\sqrt{x^2-4}}$ at $x_0 = 2$, or otherwise show that f cannot have a continuous extension at x_0.

Solution:

We check if the limit $\lim_{x\to 2^+} f(x)$ exists. We have:

$$\begin{aligned}
\lim_{x\to 2^+} \frac{\sqrt{x}-\sqrt{2}+\sqrt{x-2}}{\sqrt{x^2-4}} &= \lim_{x\to 2^+}\left(\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x-2}\cdot\sqrt{x+2}}+\frac{\sqrt{x-2}}{\sqrt{x-2}\cdot\sqrt{x+2}}\right)\\
&= \lim_{x\to 2^+}\left(\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x-2}\cdot\sqrt{x+2}}\cdot\frac{\sqrt{x-2}}{\sqrt{x-2}}\cdot\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}+\frac{1}{\sqrt{x+2}}\right)\\
&= \lim_{x\to 2^+}\left(\frac{(x-2)}{(x-2)\cdot\sqrt{x+2}}\cdot\frac{\sqrt{x-2}}{\sqrt{x}+\sqrt{2}}+\frac{1}{\sqrt{x+2}}\right)\\
&= 0+\frac{1}{2}=\frac{1}{2}
\end{aligned}$$

So we can extend the function f continuously to the interval $[2,\infty[$. We define the new function as:

$$\hat{f}(x)=\begin{cases} f(x), & x>2 \\ 1/2, & x=2 \end{cases}$$

which is continuous on $[2,\infty[$.

3. **The Bisection Algorithm:** Using the intermediate value theorem and successive bisection of the interval $[0,1]$, find an interval of the length $L \leq \frac{1}{8}$ that contains a solution of the equation

$$x^3+x-1=0.$$

Solution:

We search for a root x_0 of the function $f(x)=x^3+x-1$, i.e. x_0 that satisfies $f(x_0)=0$. The bisection algorithm is an iterative routine that successively restricts the interval that contains the root x_0.

The steps of the algorithm is given below, where L is the length of the interval where we can find a root x_0.

$$\begin{aligned}
&f(0)=-1<0 \quad \text{et} \quad f(1)=1>0 && \Longrightarrow && x_0\in\,]0,1[\,, \quad L=1\\
&f\left(\tfrac{1}{2}\right)=-\tfrac{3}{8}<0 && \Longrightarrow && x_0\in\,\left]\tfrac{1}{2},1\right[\,, \quad L=\tfrac{1}{2}\\
&f\left(\tfrac{3}{4}\right)=0.172>0 && \Longrightarrow && x_0\in\,\left]\tfrac{1}{2},\tfrac{3}{4}\right[\,, \quad L=\tfrac{1}{4}\\
&f\left(\tfrac{5}{8}\right)=-0.130<0 && \Longrightarrow && x_0\in\,\left]\tfrac{5}{8},\tfrac{3}{4}\right[\,, \quad L=\tfrac{1}{8}
\end{aligned}$$

So $x_0\in\,\left]\frac{5}{8},\frac{3}{4}\right[\,=\,]0.625,0.75[$. The exact value of the root is $x_0=0.6823\ldots$ which is indeed inside the obtained interval.

4. Let the function $f\colon[0,\infty)\to\mathbb{R}$ be defined as

$$f(x)=\begin{cases} \dfrac{3x^2-10x+3}{x^2-2x-3}\,, & x>3 \\ \alpha\,, & x=3 \\ \beta x-4\,, & x<3 \end{cases}$$

Find $\alpha, \beta \in \mathbb{R}$ such that the function is continuous at $x = 3$.

Solution:
For the function to be continuous we require $\lim_{x\to 3^-} f = \lim_{x\to 3^+} f = f(3)$We first compute the right limit at $x = 3$:

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{3x^2 - 10x + 3}{x^2 - 2x - 3} = \lim_{x\to 3^+} \frac{(x-3)(3x-1)}{(x-3)(x+1)} = 2$$

This implies that $f(3) = \alpha = 2$. Also for the limit from the left we have:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \beta x - 4 = 3\beta - 4$$

So $3\beta - 4 = 2$, and we get $\beta = 2$.

5. Show that if $f(x)$ is continuous on $[-1, 1]$ and $f(-1) = f(1)$, then there exists $\delta \in [0, 1]$ such that $f(\delta) = f(\delta - 1)$.

Solution:
First, consider the function $g(x) = f(x-1) - f(x)$, which is continuous on $[0, 1]$. Now we know the following:

$$g(0) = f(-1) - f(0)$$

and

$$g(1) = f(0) - f(1)$$

But we know that $f(-1) = f(1)$, so we can manipulate the second equation to get

$$g(1) = f(0) - f(1) = f(0) - f(-1) = -g(0)$$

- If $g(0) = 0$, then $f(-1) = f(0)$ by the first equation. Hence, $\delta = 0 \in [0, 1]$.
- If $g(0) > 0$, then $g(1) < 0$ so by the intermediate value theorem there exists a $\delta \in [0, 1]$ such that $g(\delta) = 0$, meaning $f(\delta - 1) = f(\delta)$.
- If $g(0) < 0$, then $g(1) > 0$ so by the intermediate value theorem there exists a $\delta \in [0, 1]$ such that $g(\delta) = 0$, meaning $f(\delta - 1) = f(\delta)$.

6. Find, if it exists, continuous extension of the function $f : [-\pi/4, 0[\cup]0, \pi/4] \to \mathbb{R}$ given by $f(x) = \frac{1-\cos x}{\tan^2 x}$ at $x_0 = 0$, or otherwise show that f cannot have a continuous extension at x_0.

Solution:
We check if the limit of f exists as $x \to 0$. We have

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{\tan^2 x} &= \lim_{x\to 0} \frac{1-\cos x}{\frac{\sin^2 x}{\cos^2 x}} \cdot \frac{1+\cos x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{\sin^2 x} \cdot \frac{\cos^2 x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{\sin^2 x} \cdot \frac{\cos^2 x}{1+\cos x} \\
&= \frac{1}{2}
\end{aligned}$$

So we can extend the function f continuously to the interval $[-\pi/4, \pi/4]$. We define the new function as:

$$\hat{f}(x) = \begin{cases} f(x), & x \neq 0 \\ 1/2, & x = 0 \end{cases}$$

which is continuous on $[-\pi/4, \pi/4]$.

7. Let us define the functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

(a) Find domain and range for each of the 3 functions.

(b) Show that

$$\cosh(x)^2 - \sinh(x)^2 = 1.$$

(c) Find a suitable domain, for each of the 3 functions, over which the function is invertible.

(d) Compute

$$\lim_{x\to+\infty} \cosh(x), \quad \lim_{x\to-\infty} \cosh(x),$$
$$\lim_{x\to+\infty} \sinh(x), \quad \lim_{x\to-\infty} \sinh(x),$$
$$\lim_{x\to+\infty} \tanh(x), \quad \lim_{x\to-\infty} \tanh(x).$$

Solution:

(a) As e^x, e^{-x} are defined over all of the real numbers, that is, $Dom(e^x) = Dom(e^{-x}) = \mathbb{R}$, then the same holds true for $\sinh(x), \cosh(x)$. Moreover, $\cosh(x) > 0$, $\forall x \in \mathbb{R}$, hence also $Dom(\tanh(x)) = \mathbb{R}$.
Let us examine the ranges. Let us notice that $\cosh(x)$ is even, while $\sinh(x)$ is odd, and $\tanh(x)$ is odd, as well.
$\sinh(x)$ is strictly increasing on the interval $[0, +\infty[$: in fact, taking $s > t > 0$, then $e^s > e^t$ (or, equivalently, $e^s - e^t > 0$), $e^{-s} < e^{-t}$ (or, equivalently, $e^{-s} - e^{-t} < 0$), so that

$$\sinh(s) > \sinh(t) \iff e^s - e^{-s} > e^t - e^{-t} \iff e^s - e^t > e^{-s} - e^{-t}.$$

But finally, $e^{-s} - e^{-t} < 0$, while $e^s - e^t > 0$, which show that the last inequality above indeed holds.

(b) By the definitions,

$$\cosh(x)^2 = \frac{(e^x + e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} + 2e^x e^{-x}}{4} = \frac{e^{2x} + e^{-2x} + 2e^{x-x}}{4} = \frac{e^{2x} + e^{-2x} + 2}{4},$$
$$\sinh(x)^2 = \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} - 2e^x e^{-x}}{4} = \frac{e^{2x} + e^{-2x} - 2e^{x-x}}{4} = \frac{e^2 x + e^{-2x} - 2}{4},$$
$$\implies \cosh(x)^2 - \sinh(x)^2 = \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} = \frac{4}{4} = 1.$$

(c) As $\cosh(x) = \sqrt{1+\sinh(x)^2}$ and over the positive real numbers the functions $f(x) = \sinh(x)$, $g(x) = 1 + x^2$, $h(x) = \sqrt{x}$ are all strictly increasing and $\cosh(x) = h(g(f(x)))$, then $cosh(x)$ is strictly increasing on $[0, +\infty[$. As $\cosh(x)$ is even, this is the largest interval over which the function is injective (hence, invertible when we take the arrival set to coincide with $R(\cosh(x)) = [1, +\infty[$, since $\cosh(0) = 1$ and $\lim_{x\to+\infty} \cosh(x) = +\infty$, and, again, $\cosh(x)$ is even).
As $\sinh(x)$ is odd, and strictly increasing on $[0, +\infty[$, then it is strictly increasing over all of $\mathbb{R}$ and it is therefore invertible on $\mathbb{R}$ and $R(f) = \mathbb{R}$, since $\sinh(0) = 0$ and $\lim_{x\to+\infty} \sinh(x) = +\infty$, and, again, $\sinh(x)$ is odd.

(d) Let us note that $\cosh(x) \geq |x|$, $\forall x \in \mathbb{R}$. Hence, the squeeze theorem implies immedaitely that

$$\lim_{x\to+\infty} \cosh(x) = +\infty = \lim_{x\to-\infty} \cosh(x).$$

Since $\lim_{x\to+\infty} e^x = +\infty$, then $\lim_{x\to+\infty} e^{-x} = 0$ and

$$\lim_{x\to+\infty} \frac{\sinh(x)}{e^x} = \lim_{x\to+\infty} \frac{\frac{e^x-e^{-x}}{2}}{e^x} = \lim_{x\to+\infty} \frac{\frac{e^x}{e^x} - \frac{e^{-x}}{e^x}}{2} = \frac{1}{2}.$$

Hence $\sinh(x) \geq \frac{1}{3}e^x$[1] for $x \gg 0$, hence

$$\lim_{x\to+\infty} \sinh(x) = +\infty.$$

As $\sinh(x)$ is odd, then

$$\lim_{x\to-\infty} \sinh(x) = -\infty.$$

Finally,

$$\lim_{x\to+\infty} \tanh(x) = \lim_{x\to+\infty} \frac{\sinh(x)}{\cosh(x)} = \lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} =$$
$$\lim_{x\to+\infty} \frac{e^{-x}\cdot(e^x - e^{-x})}{e^{-x}\cdot(e^x + e^{-x})} = \lim_{x\to+\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1,$$

since $\lim_{x\to+\infty} e^{-x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$.
As $\tanh(x)$ is odd, then

$$\lim_{x\to-\infty} \tanh(x) = -1.$$

8. Calculate the derivative f' of the function f and give the domain of f and f'.

(a) $f(x) = \dfrac{5x+2}{3x^2-1}$

(b) $f(x) = \tan(x)$

(c) $f(x) = x\sin(x) + \frac{\cos(x)^2}{x^2+2}$

Solution:

(a) We use the quotient rule to obtain $f'(x) = \dfrac{5(3x^2-1) - 6x(5x+2)}{(3x^2-1)^2} = -\dfrac{15x^2+12x+5}{(3x^2-1)^2}$;
$D(f) = D(f') = \mathbb{R} \setminus \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$

(b) We apply the formula for the derivative of a quotient on $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$, to obtain

$$f'(x) = \frac{\cos(x)^2 - \sin(x)\cdot\big(-\sin(x)\big)}{\cos(x)^2} = \frac{1}{\cos(x)^2}$$

So $D(f) = D(f') = \mathbb{R} \setminus \{x \in \mathbb{R} : \cos(x) = 0\} = \mathbb{R} \setminus \left\{\frac{(2k+1)\pi}{2} \ : \ k \in \mathbb{Z}\right\}$.

(c) We use the rules for product sum and quotient of derivatives.

$$(x\sin(x))' = x\cos(x) + \sin(x),$$
$$(\cos(x)^2)' = (\cos(x)\cos(x))' = \cos(x)(-\sin(x)) - \sin(x)\cos(x) = -2\sin(x)\cos(x) = -\sin(2x),$$
$$(x^2+2)' = 2x,$$
$$\begin{aligned} f'(x) &= x\cos(x) + \sin(x) + \frac{(x^2+2)(-\sin(2x)) - 2x\cos(x)^2}{(x^2+2)^2} \\ &= x\cos(x) + \sin(x) - \frac{\sin(2x)}{x^2+2} - \frac{2x\cos(x)^2}{(x^2+2)^2} \end{aligned}$$

So $D(f) = D(f') = \mathbb{R}$.

9. Prove the quotient rule for derivatives:
if $f : I \to \mathbb{R}$, $f(x) = \frac{g(x)}{h(x)}$, $x_0 \in I$ and both g and h are differentiable at x_0, with $h(x_0) \neq 0$, then, $f'(x_0) = \frac{g'(x_0)h(x_0) - g(x_0)h'(x_0)}{h(x_0)^2}$.

Solution:
For all $x_0 \in I \setminus \{x : h(x) = 0\}$, we have

$$\begin{aligned} f'(x_0) &= \lim_{s\to 0} \frac{f(x_0+s) - f(x_0)}{s} \\ &= \lim_{s\to 0} \frac{\frac{g(x_0+s)}{h(x_0+s)} - \frac{g(x_0)}{h(x_0)}}{s} \\ &= \lim_{s\to 0} \frac{g(x_0+s)h(x_0) - g(x_0)h(x_0+s)}{sh(x_0)h(x_0+s)}. \end{aligned}$$

Here we are relying on the continuity of h: indeed, if $h(x_0) \neq 0$, by continuity, $h(x_0+s) \neq 0$ for s small (cf. Exercise 2 in this worksheet).
Since g, h are differentiable at x_0, then they are also continuous at that point, hence $\lim\limits_{s\to 0} \frac{1}{h(x)h(x+s)} = \frac{1}{h(x)^2}$. Moreover,

$$\begin{aligned} &\lim_{s\to 0} \frac{g(x_0+s)h(x_0) - g(x_0)h(x_0+s)}{s} \\ &= \lim_{s\to 0} \frac{g(x_0+s)h(x_0) - g(x_0)h(x_0) + g(x_0)h(x_0) - g(x_0)h(x_0+s)}{s} \\ &= \lim_{s\to 0} \left(h(x_0)\frac{g(x_0+s) - g(x_0)}{s} - g(x_0)\frac{h(x_0+s) - h(x_0)}{s}\right) \\ &= \left(g'(x_0)h(x_0) - g(x_0)h(x_0)\right). \end{aligned}$$

Hence,

$$f'(x_0) = \frac{g'(x_0)h(x_0) - g(x_0)h(x_0)}{h(x_0)^2}.$$

10. For each of the following functions, find the inverse function and the derivative of the inverse function.

   (a) $f(x) = \cos x$, $x \in]0, \pi[$.
   (b) $f(x) = \tan x$, $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

Solution:

(a) The inverse function is given by $g(x) = f^{-1}(x) = \arccos(x)$. Since $f'(x) = -\sin x$,

$$f'(g(x)) = -\sin(\arccos(x))$$

To find $\sin(\arccos(x))$, let $\theta = \arccos(x)$. We want to find $\sin\theta$. We have

$$\theta = \arccos(x) \Rightarrow \cos\theta = x \Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-x^2},$$

where in the last step we used that $\theta \in]0, \pi[$, which guarantees that $\sin\theta > 0$. Putting everything together

$$g'(x) = \frac{1}{f'(g(x))} = -\frac{1}{\sqrt{1-x^2}}.$$

(b) The inverse function is given by $g(x) = f^{-1}(x) = \arctan(x)$. Since $f'(x) = \frac{1}{\cos^2 x}$,

$$f'(g(x)) = \cos^2(\arctan(x))$$

To find $\cos^2(\arctan(x))$, let $\theta = \arctan(x)$. We want to find $\cos\theta$. We have

$$\begin{aligned}
\theta = \arctan(x) \Rightarrow \tan\theta = x &\Rightarrow \frac{\sin\theta}{\cos\theta} = x \\
&\Rightarrow \frac{\sin^2\theta}{\cos^2\theta} = x^2 \\
&\Rightarrow \frac{\sin^2\theta + \cos^2\theta - \cos^2\theta}{\cos^2\theta} = x^2 \\
&\Rightarrow \frac{1}{\cos^2\theta} - 1 = x^2 \\
&\Rightarrow \cos^2\theta = \frac{1}{1+x^2}
\end{aligned}$$

Putting everything together

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1+x^2}.$$

11. Calculate the derivative f' of the function f and give the domain of f and f'.

   (a) $f(x) = \dfrac{x^2}{\sqrt{1-x^2}}$
   (b) $f(x) = \sin(x)^2 \cdot \cos(x^2)$
   (c) $f(x) = \sqrt{\sin\left(\sqrt{\sin(x)}\right)}$
   (d) $f(x) = \sin(x)\log(\sin(x))e^{\cos(x)}$

Solution:

(a) We use the derivative rules of quotient and of composition of functions

$$f'(x) = \frac{2x\sqrt{1-x^2} - x^2 \frac{1}{2\sqrt{1-x^2}}(-2x)}{1-x^2} = \frac{x(2-x^2)}{(1-x^2)^{3/2}} \ ; \quad D(f) = D(f') =]-1,1[.$$

(b) We use the derivative rules of multiplication and of composition of functions

$$\begin{aligned} f'(x) &= 2\sin(x)\cos(x)\cdot\cos(x^2) + \sin(x)^2\cdot\big(-\sin(x^2)\big)\cdot 2x \\ &= 2\sin(x)\big(\cos(x)\cos(x^2) - x\sin(x)\sin(x^2)\big)\,; \qquad D(f) = D(f') = \mathbb{R}. \end{aligned}$$

(c) We use the derivative rule of composition of functions

$$\begin{aligned} f'(x) &= \frac{1}{2\cdot\sqrt{\sin\Big(\sqrt{\sin(x)}\Big)}}\cos\Big(\sqrt{\sin(x)}\Big)\frac{1}{2\cdot\sqrt{\sin(x)}}\cos(x) \\ &= \frac{\cos\Big(\sqrt{\sin(x)}\Big)\cos(x)}{4\cdot\sqrt{\sin\Big(\sqrt{\sin(x)}\Big)}\cdot\sqrt{\sin(x)}}\,. \end{aligned}$$

The domain of f is

$$D(f) = \Big\{x\in\mathbb{R} : \sin(x)\geq 0 \text{ and } \sin\Big(\sqrt{\sin(x)}\Big)\geq 0\Big\} = \bigcup_{k\in\mathbb{Z}}\big[2k\pi, (2k+1)\pi\big]\,.$$

in fact, $\sin(x)\geq 0 \iff x\in\big[2k\pi,(2k+1)\pi\big]$ and for these values, we have $\sqrt{\sin(x)}\in[0,1]$ so $\sin\Big(\sqrt{\sin(x)}\Big)\geq 0$, which means f is well defined.

For the domain of f', we need to exclude the points where $\sin(x)=0$, so

$$D(f') = \bigcup_{k\in\mathbb{Z}}\big]2k\pi,(2k+1)\pi\big[\,.$$

(d)

$$\begin{aligned} f'(x) &= \cos(x)\log(\sin(x))e^{\cos(x)} + \sin(x)\frac{1}{\sin(x)}\cos(x)e^{\cos(x)} \\ &\quad + \sin(x)\log(\sin(x))e^{\cos(x)}(-\sin(x)) \\ &= e^{\cos(x)}(\log(\sin(x))(\cos(x)-\sin^2(x)) + \cos(x)). \end{aligned}$$

So $D(f) = D(f') = \bigcup_{k\in\mathbb{Z}}]2k\pi,(2k+1)\pi[$, because $\log(x)$ is defined only for $x>0$.

12. For $x\in\mathbb{R}$, e^x has been defined as $\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Hence, this definition gives rise to a function $f\colon\mathbb{R}\to\mathbb{R}$, $f(x)=e^x$. Prove the following properties of the exponential e^x:

(a) $e^0 = 1$;

(b) $e^x\cdot e^y = e^{x+y}$; [For this part of the exercise you can assume the following result:

Let $(a_n),(b_n)$ be sequences. Assume that both $\sum_{i=0}^{\infty}a_i, \sum_{i=0}^{\infty}b_i$ converge to a finite limit, and, *moreover, that at least one of* $\sum_{i=0}^{\infty} a_i, \sum_{i=0}^{\infty} b_i$ *converges absolutely. Then the sequence* (z_n), $z_n := \sum_{l=0}^{n} a_l b_{n-l}$ *satisfies*

$$\sum_{i=0}^{\infty} a_i \cdot \sum_{i=0}^{\infty} b_i = \sum_{i=0}^{\infty} z_i.$$

(c) $e^{-x} = \frac{1}{e^x}$

(d) e^x is a strictly increasing function of x; e^{-x} is a strictly decreasing function of x;

(e) Use the definition of $\log(x)$ as inverse of the function e^x to show that

(i) $\log(ab) = \log(a) + \log(b)$ for all $a, b > 0$.

(ii) $\log(a^b) = b\log(a)$ for all $a > 0$ and all $b \in \mathbb{R}$.

(iii) $\log(x)$ is a strictly increasing function of x.

Solution:

(a) $f(0) = e^0 = 1 + \frac{0}{1!} + \frac{0^2}{2!} + \cdots = 1$

(b) By D'Alembert's criterion, for any $x \in \mathbb{R}$, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent, since

$$\lim_{n\to\infty} \frac{\frac{|x^n|}{n!}}{\frac{|x^{n-1}|}{(n-1)!}} = \lim_{n\to\infty} \frac{|x|}{n} = 0.$$

Note that $\forall x, y \in \mathbb{R}$,

$$\frac{(x+y)^n}{n!} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k y^{n-k}}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}.$$

Hence, $\forall x, y \in \mathbb{R}$,

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}),$$

but the latter term is exactly equal to

$$\sum_{n=0}^{\infty} z_n, \quad \text{where } z_n := \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}.$$

The result that was cited in the text of the exercise and we are free to assume then implies that

$$\sum_{n=0}^{\infty} z_n = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right),$$

that is, $e^{x+y} = e^x \cdot e^y$.

(c) By (a), we have $e^0 = 1$. Now, fix $x \in \mathbb{R}$. By (b), we have

$$e^x \cdot e^{-x} = e^{x-x} = e^0 = 1.$$

So, this proves that $e^x \neq 0$, $e^{-x} \neq 0$, and that $e^{-x} = \frac{1}{e^x}$. As a byproduct, it follows that $e^x > 0$ for all x: indeed, this is clear from the definition if $x > 0$, as each summand $\frac{x^n}{n!} > 0$. Then, the case $x = 0$ follows from (a), while the case $x < 0$ follows from the case $x > 0$ and part (c) we just proved.

(d) If $a, b \in \mathbb{R}$ such that $a < b$, then $e^a < e^b$ holds if and only if $e^{b-a} > 1$ by using (c). But

$$e^{b-a} = \sum_{k=0}^{\infty} \frac{(b-a)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(b-a)^k}{k!} > 1$$

because $b - a > 0$. Now to show that e^{-x} is a decreasing function it suffices to note that e^{-x} is a the reciprocal of an increasing function.

(e) (i) $e^{\log(ab)} = ab = e^{\log(a)}e^{\log(b)} = e^{\log(a)+\log(b)}$. Since e^x is strictly increasing, it is injective; thus, we conclude that $\log(ab) = \log(a) + \log(b)$.

(ii) $e^{\log(a^b)} = a^b = (e^{\log(a)})^b = e^{b\log(a)}$. Since e^x is injective, we conclude that $\log(a^b) = b\log(a)$.

(iii) $\log(x)$ is the inverse function of a strictly increasing function, hence, it is strictly increasing.

13. For each function, calculate $f^{(n)}$, the n-th order derivative of f.

(a) $f(x) = x^m \quad (m \in \mathbb{Z})$

(b) $f(x) = \sin(2x) + 2\cos(x)$

(c) $f(x) = \log(x)$

Solution:

(a) We identify three cases for m:

- $m = 0$: $f^{(n)}(x) = 0$ for all $n \in \mathbb{N}^*$.
- $m \geq 1$: $f^{(n)}(x) = \begin{cases} m(m-1)(m-2)\cdots(m-n+1)x^{m-n}, & n \leq m \\ 0, & n > m \end{cases}$ The first case follows immediately, while the second and the third can be proved by induction on m (resp. $-m$).
- $m \leq -1$: $f^{(n)}(x) = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$ for all $n \in \mathbb{N}^*$

(b) We start by calculating the first four derivatives of f:

$$f'(x) = 2\cos(2x) - 2\sin(x) \qquad f''(x) = -4\sin(2x) - 2\cos(x)$$
$$f'''(x) = -8\cos(2x) + 2\sin(x) \qquad f^{(4)}(x) = 16\sin(2x) + 2\cos(x)$$

We need to distinguish two cases for $n \in \mathbb{N}^*$:

$$f^{(n)}(x) = \begin{cases} (-1)^{\frac{n}{2}}\left(2^n\sin(2x) + 2\cos(x)\right), & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\left(2^n\cos(2x) - 2\sin(x)\right), & n \text{ odd} \end{cases}$$

(c) Since $f'(x) = x^{-1}$, we can use the result in part (a) with $m = -1$ to obtain $f^{(n)}$. We have,

$$f^{(n)}(x) = (f')^{(n-1)}(x) = (-1)(-2)(-3)\cdots(-(n-1))x^{-1-(n-1)} = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. State if the following are true or false.

(a) f even $\Rightarrow$ f' odd,

(b) f odd $\Rightarrow$ f' even,

(c) f' even $\Rightarrow$ f odd,

(d) f periodic $\Rightarrow$ f' periodic.

Solution:

(a) True. We have $f(-x) = f(x)$. Using the derivative of composite functions we obtain $-f'(-x) = f'(x)$, so f' is odd.

(b) True. We take the derivative of $f(-x) = -f(x)$ to obtain $-f'(-x) = -f'(x) \Leftrightarrow f'(-x) = f'(x)$. So f' is even.

(c) False. Take for example $f(x) = x + 1$.

(d) True. For a periodic function f, there exists $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. By taking the derivative, we have $f'(x+T) = f'(x)$ and so f' is also periodic.

15. Calculate $(g \circ f)'(0)$ for the functions $f, g \colon \mathbb{R} \to \mathbb{R}$ defined by

(a) $f(x) = 2x + 3 + (e^x - 1)\sin(x)^7\cos(x)^4$ and $g(x) = \log(x)^3$.

(b) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + 2x\,, & x \neq 0 \\ 0\,, & x = 0 \end{cases}$ and $g(x) = (x-1)^4$.

Solution: Since $(g \circ f)'(0) = g'(f(0)) \cdot f'(0)$, we need to find the derivatives of f and g.

(a) To find $f'(x)$, we write $f(x) = 2x+3+(e^x - 1)\,u(x)$ where $u(x) = \sin(x)^7\cos(x)^4$. Then

$$f'(x) = 2+e^x\,u(x)+(e^x-1)\,u'(x) \quad \text{and} \quad u'(x) = 7\sin(x)^6\cos(x)^5 - 4\sin(x)^8\cos(x)^3.$$

We have $u(0) = u'(0) = 0$ and so $f'(0) = 2$.

Then we have $g'(x) = \dfrac{3\log(x)^2}{x}$. Since $f(0) = 3$ we finally show that

$$(g \circ f)'(0) = g'(3) \cdot f'(0) = \frac{3\log(3)^2}{3} \cdot 2 = 2\log(3)^2.$$

(b) For calculating $f'(0)$, we must use the definition of derivative:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) + 2x - 0}{x} = \lim_{x\to 0}\left(x\sin\left(\tfrac{1}{x}\right) + 2\right) = 2$$

This is because $\lim_{x\to 0}\left(x \sin\left(\frac{1}{x}\right)\right) = 0$, as the function sin is bounded, and so $x\sin\left(\frac{1}{x}\right)$ is squeezed to 0 by the factor x as $x \to 0$.

Since $g'(x) = 4(x-1)^3$ and $f(0) = 0$, we obtain

$$(g \circ f)'(0) = g'(0) \cdot f'(0) = (-4) \cdot 2 = -8.$$

16. Calculate the derivative f' of the function f and give the domain of f and f'.

   (a) $f(x) = \sqrt[5]{\left(2x^4 + e^{-(4x+3)}\right)^3}$
   (b) $f(x) = e^{\sqrt[3]{\log(4x)^2}}$
   (c) $f(x) = \log\left(4^{\sin(x)}\right)e^{\cos(4x)}$

Solution:

(a) $f'(x) = \frac{3}{5}\left(2x^4 + e^{-(4x+3)}\right)^{-2/5}\left(8x^3 - 4e^{-(4x+3)}\right) = \dfrac{12\left(2x^3 - e^{-(4x+3)}\right)}{5\sqrt[5]{\left(2x^4 + e^{-(4x+3)}\right)^2}}$;

$D(f) = D(f') = \mathbb{R}$ (The denominator of f' is nonzero since $e^{-(4x+3)} > 0$ and $x^4 \geq 0$ for all $x \in \mathbb{R}$.)

(b)

$$f'(x) = e^{\sqrt[3]{\log^2(4x)}}\frac{2}{3}(\log(4x))^{-\frac{1}{3}}\frac{1}{4x} \cdot 4 = \frac{2}{3}\frac{e^{\sqrt[3]{\log^2(4x)}}}{x\sqrt[3]{\log(4x)}}$$

The domain of f is $]0, +\infty[$ because $\log(x)$ is defined only for $x > 0$. For the domain of f' we have to exclude all the points where the denominator vanishes, that is $x = \frac{1}{4}$, because $\log(4x) = 0$ if and only if $4x = 1$. So $D(f') =]0, \frac{1}{4}[\cup]\frac{1}{4}, +\infty[$.

(c) We have $f(x) = \sin(x)\log(4)\,e^{\cos(4x)}$ by Exercise 5. We obtain

$$\begin{aligned} f'(x) &= \log(4)\cos(x)\,e^{\cos(4x)} + \log(4)\sin(x) \cdot \big(-4\sin(4x)\big) \cdot e^{\cos(4x)} \\ &= \log(4)\,e^{\cos(4x)}\Big(\cos(x) - 4\sin(x)\sin(4x)\Big). \end{aligned}$$

$D(f) = D(f') = \mathbb{R}$

17. State if the following are true or false.

   (a) If f is differentiable at $a \in \mathbb{R}$, Then there is $\delta > 0$ such that f is continuous on $]a-\delta, a+\delta[$.
   (b) If f is differentiable from left and right at $a \in \mathbb{R}$, then f is differentiable at a.
   (c) If f is differentiable on $\mathbb{R}$, then $g(x) = \sqrt{f^2(x)}$ is differentiable on $\mathbb{R}$.

Solution:

(a) False. Take for example $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0\,, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. This function is continuous at $x = 0$ because,

$$0 \leq f(x) \leq x^2$$

for all $x \in \mathbb{R}$. So by the squeeze theorem we have $\lim_{x \to 0} f(x) = 0 = f(0)$. On the other hand f is not continuous at any point other than 0. In fact, let $x_0 \in \mathbb{R}$, $x_0 \neq 0$. For $n \in \mathbb{N}^*$, The open interval $\left]x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right[$ contains rational and irrational numbers. We take $a_n \in \left]x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right[\cap \mathbb{Q}$ and $b_n \in \left]x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right[\cap (\mathbb{R} \setminus \mathbb{Q})$ for each $n \in \mathbb{N}^*$. Then the two sequences $(a_n) \subset \mathbb{Q}$ and $(b_n) \subset \mathbb{R} \setminus \mathbb{Q}$ both converge to x_0, but

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n^2 = x_0^2 > 0 = \lim_{n \to \infty} f(b_n),$$

so f is not continuous at x_0.

But f is differentiable at $x = 0$. Indeed, we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

and since $-|x| \leq \frac{f(x)}{x} \leq |x|$ for all $x \in \mathbb{R}$, the squeeze theorem gives

$$f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = 0\,.$$

(b) False. Take for example $f(x) = |x|$ which is not differentiable at 0. But the left and right derivatives exist (Look at the lecture notes).

(c) False. By taking $f(x) = x$, we have $g(x) = \sqrt{x^2} = |x|$ which is not differentiable at 0.

18. For each of the following functions, find the inverse function. Find the derivative of the inverse function once by direct calculation and once by the inverse function derivative.

(a) $f(x) = \sqrt{x^2 + 9}$, $x \geq 0$.

(b) $f(x) = \frac{1}{1+x}$, $x \neq -1$.

Solution:

(a) To find $g(x) = f^{-1}$, solve $y = \sqrt{x^2 + 9}$ for x. This yields $x = \pm\sqrt{y^2 - 9}$. Because the domain of f is restricted to $x \geq 0$, we must choose the positive sign in front of the radical. Thus

$$g(x) = f^{-1}(x) = \sqrt{x^2 - 9}$$

By the formula for the derivative of the inverse function we have

$$g'(x) = \frac{1}{f'(g(x))}$$

with

$$f'(x) = \frac{x}{\sqrt{x^2 + 9}}$$

so

$$f'(g(x)) = \frac{\sqrt{x^2 - 9}}{\sqrt{(\sqrt{x^2 - 9})^2 + 9}} = \frac{\sqrt{x^2 - 9}}{\sqrt{x^2}} = \frac{\sqrt{x^2 - 9}}{x}$$

since the domain of g is $x \geq 3$. Thus,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{x}{\sqrt{x^2 - 9}}$$

This agrees with the answer we obtain by differentiating directly:

$$g'(x) = \frac{2x}{2\sqrt{x^2 - 9}} = \frac{x}{\sqrt{x^2 - 9}}$$

(b) To find $g(x) = f^{-1}$, solve $y = \frac{1}{1+x}$ for x. This yields $x = \frac{1-y}{y}$. Thus

$$g(x) = \frac{1-x}{x}$$

By direct calculation we can rewrite $g(x) = x^{-1} - 1$. So we see that $g'(x) = -x^{-2}$. Also we see that $f'(x) = -(1+x)^{-2}$, so

$$f'(g(x)) = -\left(1 + \frac{1-x}{x}\right)^{-2} = -(x^{-1})^{-2} = -x^2$$

and

$$g'(x) = \frac{1}{f'(g(x))} = -x^{-2}.$$

19. Find maximum and minimum of the following functions
   (a) $f(x) = x^2 - 5$ in $[-\pi, \pi]$
   (b) $f(x) = \sqrt[3]{(x-1)(x-2)^2}$ in $[1 + \frac{1}{10}, 2 - \frac{1}{10}]$

Solution:

(a) The derivative is $2x$, and vanishes at 0. We compute

$$f(0) = -5, \quad f(-\pi) = f(\pi) > 0.$$

So the minimum is at 0 and there are two maxima, at $-\pi$ and π.

(b) The derivative is

$$f'(x) = \frac{(x-2)(3x-4)}{(3(x-1)(x-2)^2)^{\frac{2}{3}}}$$

This is zero at $x_0 = 4/3$. We compute

$$f\left(\frac{4}{3}\right) = \frac{\sqrt[3]{4}}{3}, \quad f\left(1 + \frac{1}{10}\right) = \frac{3\sqrt[3]{3}}{10}, \quad f\left(2 - \frac{1}{10}\right) = \frac{\sqrt[3]{9}}{10}.$$

So we have a maximum at $\frac{4}{3}$ and a minimum at $2 - \frac{1}{10}$.

20. Calculate f'
   (a) $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{\cos x}{2 + \sin(\log x)}$
   (b) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \log(a|x|)$, $a > 0$
   (c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x^2 \sin x}$

Solution:

(a)

$$f'(x) = \frac{-\sin(x)\ (2 + \sin(\log x)) - \frac{1}{x}\cos x \cos(\log x)}{(2 + \sin(\log x))^2}$$

(b) If $x > 0$ we have $f(x) = \log(ax)$ and so,

$$f'(x) = \frac{a}{ax} = \frac{1}{x}$$

If $x < 0$ we have $f(x) = \log(-ax)$ and so

$$f'(x) = \frac{-a}{-ax} = \frac{1}{x}$$

So, we can say that that $f'(x) = \frac{1}{x}$ for all $x \in \mathbb{R} \setminus \{0\}$

(c) $f'(x) = (2x \sin x + x^2 \cos x)e^{x^2 \sin x}$