Analysis 1 - Exercise Set 9

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Show that:
   (a) $\lim_{x\to+\infty} e^x = +\infty$;
   (b) $\lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2}$, $\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$;
   (c) $\lim_{x\to+\infty} \arctan(e^x) = +\frac{\pi}{2}$.

Solution:

(a) $f(x) = e^x$ is strictly increasing, moreover, $\lim_{n\to\infty} e^n = +\infty$. Then $\lim_{x\to+\infty} e^x = +\infty$.

(b) $\arctan(x)$ is an increasing function, since it is the inverse of an increasing function. Hence, it suffices to show that there is a sequence x_n that $\lim_{n\to\infty} x_n = +\infty$ such that $\lim \arctan(x_n) = \frac{\pi}{2}$. Let $x_n = \tan(\frac{\pi}{2} - \frac{1}{n})$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \tan(\frac{\pi}{2} - \frac{1}{n}) = \lim_{n\to\infty} \frac{\sin(\frac{\pi}{2} - \frac{1}{n})}{\cos(\frac{\pi}{2} - \frac{1}{n})} = +\infty,$$

since

$$\lim_{n\to\infty} \sin(\frac{\pi}{2} - \frac{1}{n}) = 1, \quad \lim_{n\to\infty} \cos(\frac{\pi}{2} - \frac{1}{n}) = 0,$$

and the latter limit is going to 0 from the positives, as $\frac{\pi}{2} - \frac{1}{n}$ is an angle in the first quadrant. As arctan is the inverse function of tan (at least over the domain $]-\frac{\pi}{2}, \frac{\pi}{2}[$), then $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $\arctan(\tan(x)) = x$. Then $\arctan(x_n) = \arctan(\tan(\frac{\pi}{2} - \frac{1}{n})) = \frac{\pi}{2} - \frac{1}{n}$. Thus,

$$\lim_{n\to\infty} \arctan(x_n) = \lim_{n\to\infty} \frac{\pi}{2} - \frac{1}{n} = \frac{\pi}{2}.$$

For the limit $\lim_{x\to-\infty} \arctan(x)$, we argue analogously with the sequence defined as $y_n = \tan(-\frac{\pi}{2} + \frac{1}{n})$.

(c) To show that the limit converges to $\frac{\pi}{2}$, it suffices to show that for any sequence (x_n) such that $\lim_{n\to\infty} x_n = +\infty$, then $\lim_{n\to\infty} \arctan(e^{x_n}) = +\frac{\pi}{2}$. Since $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{x\to+\infty} e^x = +\infty$ the sequence $y_n = e^{x_n}$ diverges and $\lim_{n\to\infty} y_n = +\infty$. In turn, this last limit and the fact that $\lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2}$ together imply that $\lim_{n\to\infty} \arctan(y_n) = +\frac{\pi}{2}$. But $\arctan(y_n) = \arctan(e^{x_n})$ so the proof terminates.

2. Make an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous only at one point. Can you generalize this to make an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous only at n points for fixed $n \in \mathbb{N}$?

Solution:

The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & x \in \mathbb{Q}, \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

is continuous at 0. Because $\forall \varepsilon > 0 \quad \exists \delta > 0$ (take $\delta = \varepsilon$) s.t. $|x| < \delta \Rightarrow |f(x)| \leq |x| < \varepsilon$.

It is not continuous at any other point. Take any $q \in \mathbb{Q}$. Let $\varepsilon = \frac{|q|}{2}$. Then $\forall \delta > 0 \quad \exists x \in [q - \delta, q + \delta]$ s.t. $|f(x) - f(q)| \geq \varepsilon$. Take any irrational $x \in [q - \delta, q + \delta]$, then $|f(x) - f(q)| = |q| \geq \frac{|q|}{2}$. Thus, f is not continuous at any non-zero rational number.

Take any irrational number $x \in \mathbb{R}$. Let $\varepsilon = \frac{|x|}{2}$. Then $\forall \delta > 0 \quad \exists y \in [x - \delta, x + \delta]$ s.t. $|f(y) - f(x)| = |f(y)| \geq \varepsilon = \frac{|x|}{2}$. Any rational $y \in (x, x + \delta)$ will satisfy this.

A function that is continuous at only the points $a_1, \cdots, a_n \in \mathbb{R}$ is the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) := \begin{cases} p(x) & x \in \mathbb{Q}, \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

where $p(x)$ is defined as follows. We may order the the numbers $a_1 < a_2 < \cdots < a_n$ by reindexing. For $i = 1, \cdots, n - 1$ let $b_1 \in (a_i, a_{i+1})$ be the midpoint of $\{a_i, a_{i+1}\}$. Then, define

$$p(x) := \begin{cases} x - a_i & x \in [b_{i-1}, b_i) \quad \text{for } i = 2, \cdots, n, \\ x - a_1 & x < b_1, \\ x - a_n & x \geq b_n \end{cases}$$

In each interval, $g(x)$ is a shifted version of $f(x)$ and the continuity of g at $a_1, \cdots, a_n$ follows from the continuity of f at 0. Similarly, the discontinuity of g at any other point follows from the discontinity at any non-zero point of f.

3. Use the ϵ, δ definition of continuity to show that $f(x) := \sin(x)$ is continuous everywhere. (Hint: Using geometry, show $\sin(\theta) \leq \theta \quad \forall \theta \in [0, \frac{\pi}{2}]$. Then, using this result show that $|\sin(\theta)| \leq |\theta| \quad \forall \theta \in \mathbb{R}$.)

Solution:

Let $\theta \in [0, \frac{\pi}{2}]$. Consider the arc of the unit circle parametrisized by $\{(\cos(s), \sin(s)) : 0 \leq s \leq \theta\}$. The length of this arc by $L(\theta) = \theta$. Note that the shortest path from the point $(1, 0)$ and $(\cos(\theta), \sin(\theta))$ is a straight line. This line has length $l = \sqrt{(1 - \cos(\theta))^2 + \sin(\theta)^2} \geq \sin(\theta)$ and since this is the shortest path between $(1, 0)$ and $(\cos(\theta), \sin(\theta))$ we have $\theta = L(\theta) \geq l \geq \sin(\theta)$. Thus, $\sin(\theta) \leq \theta \quad \forall \theta \in [0, \frac{\pi}{2}]$. Now let $\theta > \frac{\pi}{2}$, then $|\sin(\theta)| \leq 1 < \frac{\pi}{2} < \theta$. Hence, $|\sin(\theta)| \leq \theta \quad \forall \theta \geq 0$. To show that $|\sin(\theta)| \leq |\theta| \quad \forall \theta \in \mathbb{R}$ it suffices to note that $|\sin(\theta)| \leq |\theta| \Rightarrow |\sin(-\theta)| \leq |-\theta|$.

Let $x_0 \in \mathbb{R}$. Now, let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Then, if $|x - y| < \delta = \varepsilon$ we have $|\sin(x_0) - \sin(x)| = 2|\cos(\frac{x+x_0}{2})\sin(\frac{x-x_0}{2})| \leq 2|\sin(\frac{x-y}{2})| \leq |x - y| \leq \delta = \varepsilon$. Hence, $\sin(x)$ is continuous, as required.

4. We say that a function f defined on a pointed neighborhood E of $x_0 \in \mathbb{R}$ has a continuous extension at the point x_0 if there exists a number $a \in \mathbb{R}$ such that the function

$$\hat{f}(x) := \begin{cases} f(x) & \text{if } x \in E, \\ a & \text{if } x = x_0, \end{cases}$$

is continuous at x_0.

Find, if it exists, a continuous extension of the function $f\colon [0,1[\cup]1,\infty[\to \mathbb{R}$,

$$f(x) = \frac{\sqrt{x+1}-\sqrt{2x}}{\sqrt{1+2x}-\sqrt{3}}$$

in $x_0 = 1$, or otherwise show that f cannot have a continuous extension at x_0. Finally, compute $\lim_{x\to+\infty} f(x)$.

Solution:

If the limit $\lim_{x\to x_0} f(x)$ exists in $\mathbb{R}$ then we can choose it as value a to extend f at x_0. We compute the limit of f at $x_0 = 1$. For $x \neq 1$, we can write, using the fact that $a^2 - b^2 = (a-b)(a+b)$:

$$\lim_{x\to1} f(x) = \lim_{x\to1} \frac{\sqrt{x+1}-\sqrt{2x}}{\sqrt{1+2x}-\sqrt{3}} = \lim_{x\to1} \left(\frac{\sqrt{x+1}-\sqrt{2x}}{\sqrt{1+2x}-\sqrt{3}} \cdot \frac{\sqrt{x+1}+\sqrt{2x}}{\sqrt{1+2x}+\sqrt{3}} \cdot \frac{\sqrt{1+2x}+\sqrt{3}}{\sqrt{x+1}+\sqrt{2x}} \right)$$

$$= \lim_{x\to1} \left(\frac{1-x}{2(x-1)} \cdot \frac{\sqrt{1+2x}+\sqrt{3}}{\sqrt{x+1}+\sqrt{2x}} \right) = -\frac{1}{2} \lim_{x\to1} \frac{\sqrt{1+2x}+\sqrt{3}}{\sqrt{x+1}+\sqrt{2x}} = -\frac{1}{2} \cdot \frac{2\sqrt{3}}{2\sqrt{2}} = -\frac{\sqrt{6}}{4},$$

because polynomial functions are continuous everywhere and $\sqrt{x}$ is continuous for $x > 0$ and the composition of continuous functions is continuous. So a continuous extension of f is

$$\hat{f}_1\colon [0,\infty[\longrightarrow \mathbb{R}, \qquad \hat{f}_1(x) = \begin{cases} \dfrac{\sqrt{x+1}-\sqrt{2x}}{\sqrt{1+2x}-\sqrt{3}}, & x \neq 1 \\[2ex] -\dfrac{\sqrt{6}}{4}, & x = 1 \end{cases}$$

By Algebra of Limits

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\sqrt{x+1}}{\sqrt{1+2x}-\sqrt{3}} - \lim_{x\to+\infty} \frac{\sqrt{2x}}{\sqrt{1+2x}-\sqrt{3}} = \lim_{x\to+\infty} \frac{1}{\frac{\sqrt{1+2x}-\sqrt{3}}{\sqrt{x+1}}} - \lim_{x\to+\infty} \frac{1}{\frac{\sqrt{1+2x}-\sqrt{3}}{\sqrt{2x}}} = \frac{1}{\sqrt{2}} - 1$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} \sin(x) & \text{if } x \in \mathbb{Q} \\ \cos(x) & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

State whether the following sentences are true or false.

(a) f is bounded.

(b) $\max\{f(x) : x \in [0, 2\pi[\} = 1$.

(c) $\min\{f(x) : x \in [0, 2\pi[\} = -1$.

(d) $\lim_{x \to \frac{\pi}{2}} f(x) = 1$.

(e) f is continuous.

(f) f is continuous at $x_0 = \frac{\pi}{4}$.

Solution:

(a) True, because $-1 \leq \sin(x) \leq 1$ for all $x \in \mathbb{R}$ and $-1 \leq \cos(x) \leq 1$ for all $x \in \mathbb{R}$.

(b) False. Indeed, $\sin(x)$ attains the value 1 in the interval $[0, 2\pi[$ only at $x = \frac{\pi}{2}$. But $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0 \neq 1$. Similarly, $\cos(x)$ attains the value 1 in the interval $[0, 2\pi[$ only at $x = 0$. But $f(0) = \sin(0) = 0 \neq 1$. So $1 \notin \{f(x) : x \in [0, 2\pi[\}$.

(c) True. Indeed, $f(x) \geq -1$ for all $x \in [0, 2\pi[$ and $f(\pi) = \cos(\pi) = -1$.

(d) False. We now find two sequence that converge to $\frac{\pi}{2}$ such that the sequences obtained by evaluating the function converge to distinct limit values. Let $x_n = \frac{\pi}{2} + \frac{1}{n}$, then (x_n) is a sequence that converges to $\frac{\pi}{2}$ and

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \cos\left(\frac{\pi}{2} + \frac{1}{n}\right) = 0.$$

For every $n \in \mathbb{N} \setminus \{0\}$ let y_n be a rational number such that $\frac{\pi}{2} \leq y_n \leq \frac{\pi}{2} + \frac{1}{n}$. Then (y_n) is a sequence that converges to $\frac{\pi}{2}$ (by squeeze theorem), and

$$\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} \sin(y_n) = 1.$$

(e) False. If f were continuous, then the limit $\lim_{x \to \frac{\pi}{2}} f(x)$, would exist. But it does not, as we have seen in the previous part.

(f) True. Indeed, $f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos(.), \sin(.) : \mathbb{R} \to \mathbb{R}$ are both continuous functions. So for every $\varepsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that for $|x - \frac{\pi}{4}| \leq \delta_1$ then $|\cos(x) - \frac{\sqrt{2}}{2}| \leq \varepsilon$ and for $|x - \frac{\pi}{4}| \leq \delta_2$ then $|\sin(x) - \frac{\sqrt{2}}{2}| \leq \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$, then for $|x - \frac{\pi}{4}| \leq \delta$ we have $|f(x) - \frac{\sqrt{2}}{2}| \leq \varepsilon$.

6. Compute the following limits if they exist.

(a) $\lim_{x \to 1} \frac{x^3 + x^2 - 2}{x - 1}$

(b) $\lim_{x \to 2} \frac{(-1)^{[x]} x^2 + 3}{x - 2}$

(c) $\lim_{x \to 0} \frac{\sin(x)^2}{\sin(x^2)}$

Solution:

(a) We observe that the denominator divides the numerator. So,

$$\lim_{x \to 1} \frac{x^3 + x^2 - 2}{x - 1} = \lim_{x \to 1} \frac{(x - 1)\left(x^2 + 2x + 2\right)}{x - 1} = \lim_{x \to 1} \left(x^2 + 2x + 2\right).$$

Finally we write:

$$\lim_{x\to 1}\left(x^2+2x+2\right)=\lim_{x\to 1}x^2+\lim_{x\to 1}2x+2=\left(\lim_{x\to 1}x\right)^2+2\lim_{x\to 1}x+2=1^2+2\cdot 1+2=5.$$

(b) We can assume that x belongs to the pointed neighborhood $]1,3[$ of 2. We observe that $[x]=2$ for $x\in]2,3[$ and $[x]=1$ for $x\in]1,2[$. So we compute the left and right limits at 2.

$$\lim_{x\to 2^+}\frac{(-1)^{[x]}x^2+3}{x-2}=\lim_{x\to 2^+}\frac{(-1)^2x^2+3}{x-2}=\lim_{x\to 2^+}(x^2+3)\cdot\frac{1}{x-2}=+\infty$$

because $\lim_{x\to 2^+}x^2+3$ exists in $\mathbb{R}$, $\lim_{x\to 2^+}\left|\frac{1}{x-2}\right|=+\infty$ and $(x^2+3)\cdot\frac{1}{x-2}>0$ for all $x\in]2,3[$.

$$\lim_{x\to 2^-}\frac{(-1)^{[x]}x^2+3}{x-2}=\lim_{x\to 2^-}\frac{-x^2+3}{x-2}=\lim_{x\to 2^-}(x^2-3)\cdot\frac{1}{2-x}=+\infty$$

because $\lim_{x\to 2^-}x^2-3$ exists in $\mathbb{R}$, $\lim_{x\to 2^-}\left|\frac{1}{2-x}\right|=+\infty$ and $(x^2-3)\cdot\frac{1}{2-x}>0$ for all $x\in]\sqrt{3},2[$. So the limit exists and has value $+\infty$.

(c) We have

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)^2}{\sin(x^2)}&=\lim_{x\to 0}\left(\frac{\sin(x)^2}{x^2}\cdot\frac{x^2}{\sin(x^2)}\right)\\&=\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)^2\cdot\lim_{x\to 0}\frac{x^2}{\sin(x^2)}=1^2\cdot 1=1,\end{aligned}$$

as $\lim_{x\to 0}\frac{\sin(x)}{x}=1$.

7. Let $A=\left\{\left(\frac{\pi}{2}+n\pi\right)^{-1}:n\in\mathbb{N}\right\}$. Find, if it exists, a continuous extension of the function $f\colon]0,1]\setminus A\to\mathbb{R}$,

$$f(x)=\tan\left(\frac{1}{x}\right)\left(1-\sin\left(\frac{1}{x}\right)^2\right)$$

at the points $x_0\in A\cup\{0\}$, or otherwise show that f cannot have a continuous extension at x_0.

Solution:

If the limit $\lim_{x\to x_0}f(x)$ exists in $\mathbb{R}$ then we can choose it as value a to extend f at x_0, and this is the only possible choice for a continuous extension of f at x_0. We compute the limit of f at $x_0\in A\cup\{0\}$. Note that for all $x\notin A\cup\{0\}$ we have the relation

$$\tan\left(\frac{1}{x}\right)\left(1-\sin\left(\frac{1}{x}\right)^2\right)=\tan\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)^2=\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right).$$

Let $a_n=\left(\frac{\pi}{2}+n\pi\right)^{-1}\in A$. So we have

$$\lim_{x\to a_n}f(x)=\lim_{x\to a_n}\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)=\sin\left(\frac{\pi}{2}+n\pi\right)\cos\left(\frac{\pi}{2}+n\pi\right)=(-1)^n\cdot 0=0,$$

So we can say that f indeed has a continuous extension at all $x_0 \in A$. For $x_0 = 0$ we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right) = \lim_{x\to 0^+} \frac{1}{2}\sin\left(\frac{2}{x}\right) \quad \text{which does not exist.}$$

So f cannot have a continuous extension at $x = 0$. So a continuous extension of f is

$$\hat{f}_A \colon]0,1] \longrightarrow \mathbb{R}, \qquad \hat{f}_A(x) = \begin{cases} \tan\left(\frac{1}{x}\right)\left(1 - \sin\left(\frac{1}{x}\right)^2\right), & x \notin A \\ 0, & x \in A. \end{cases}$$

8. Find the values $\alpha \in \mathbb{R}$ such that the limit $\lim_{x\to\alpha} \dfrac{x^4 - 2\alpha x^3 + 4x^2}{(x-\alpha)^2}$ exists in $\mathbb{R}$.

Solution:

If α is not a double root of the polynomial in the numerator, then the limit is $+\infty$ or $-\infty$ and hence not in $\mathbb{R}$. If α is a double root of the polynomial in the numerator, the the limit becomes a limit of a polynomial function, and hence it exists in $\mathbb{R}$ because polynomial functions are continuous. So it remains to compute the double roots of the polynomial in the numerator.

By evaluating the numerator in α we get:

$$\alpha^4 - 2\alpha^4 + 4\alpha^2 = \alpha^2(4 - \alpha^2) = \alpha^2(2+\alpha)(2-\alpha).$$

So the candidates for the roots of this polynomial are $\alpha \in \{0, -2, 2\}$.

For $\alpha = 0$, The polynomial is $x^4 + 4x^2 = x^2(x^2+4)$ so 0 is indeed a double root.

For $\alpha = \pm 2$, we have

$$x^4 \mp 4x^3 + 4x^2 = x^2(x^2 \mp 4x + 4) = x^2(x \mp 2)^2$$

So 2 and -2 are both double roots.

Thus the limit exists in $\mathbb{R}$ if and only if $\alpha \in \{-2, 0, 2\}$.

9. Study the continuity of the function $f \colon \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \dfrac{1}{1 + 2^{\frac{1}{x}}}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

at $x = 0$.

Solution:

We want to calculate the limit of $f(x)$ as $x \to 0$. For this we define a new variable u such that $x = \frac{1}{u}$:

$$\lim_{x\to 0^-} f(x) = \lim_{u\to -\infty} f\left(\tfrac{1}{u}\right) = \lim_{u\to -\infty} \frac{1}{1+2^u} = 1 = f(0),$$
$$\lim_{x\to 0^+} f(x) = \lim_{u\to +\infty} f\left(\tfrac{1}{u}\right) = \lim_{u\to +\infty} \frac{1}{1+2^u} = 0 \neq f(0).$$

So f is not continuous since the left and right limits at $x = 0$ are different.

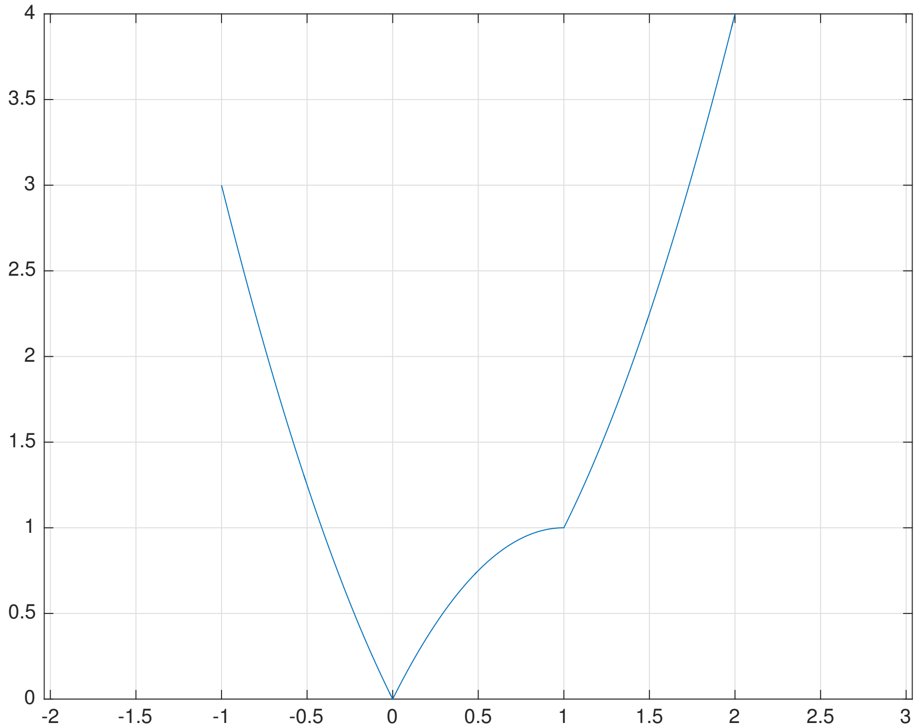
10. Find the local and global maximum/minimum of the function $f(x) = |x^2 - x| + |x|$, by sketching the graph of the function.

Solution:

We can rewrite the function as

$$f = \begin{cases} x^2 & \text{if} \quad x \geq 1 \\ -x^2 + 2x & \text{if} \quad 0 \leq x < 1 \\ x^2 - 2x & \text{if} \quad x < 0 \end{cases}$$

If we sketch the graph of this function we have



So the function attains its global minimum at $x = 0$ and has no global maximum.

11. Compute the following limits if they exist.

(a) $\lim_{x \to 1} \frac{x^2 - x}{x^2 - 2x + 1}$

(b) $\lim_{x \to +\infty} \left(\sqrt[3]{x+1} - \sqrt[3]{x} \right)$

(c) $\lim_{x \to 0} \frac{(-1)^{[x]}}{\sin(x)^3} + \frac{1}{\sin(x)^2}$

Solution:

(a) We have
$$\frac{x^2-x}{x^2-2x+1}=\frac{x(x-1)}{(x-1)^2}=\frac{x}{x-1}$$
So the limit from the left is $-\infty$, and from the right is $+\infty$. We conclude that the limit does not exist.

(b) We use the formula $a^3-b^3=(a-b)(a^2+ab+b^2)$ for $a=\sqrt[3]{x+1}$ and $b=\sqrt[3]{x}$ to obtain,
$$\lim_{x\to+\infty}\left(\sqrt[3]{x+1}-\sqrt[3]{x}\right)=\lim_{x\to+\infty}\frac{\left((x+1)^{\frac{1}{3}}-x^{\frac{1}{3}}\right)\left((x+1)^{\frac{2}{3}}+(x+1)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right)}{(x+1)^{\frac{2}{3}}+(x+1)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}}$$
$$=\lim_{x\to+\infty}\frac{1}{(x+1)^{\frac{2}{3}}+(x+1)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}}=0\,.$$

(c) We can assume that $x\in]-1,1[$. So $[x]=0$ for all $x\in]-1,1[$. We compute the right and left limits.
$$\lim_{x\to0^+}\frac{(-1)^{[x]}}{\sin(x)^3}+\frac{1}{\sin(x)^2}=\lim_{x\to0^+}\frac{1}{\sin(x)^2}\left(\frac{(-1)^{[x]}}{\sin(x)}+1\right)=\lim_{x\to0^+}\frac{1}{\sin(x)^2}\left(\frac{1}{\sin(x)}+1\right)=+\infty$$
because $\lim_{x\to0^+}\sin(x)=0$ and $\frac{1}{\sin(x)}\geq0$ for $x\in]0,1[$.
$$\lim_{x\to0^-}\frac{(-1)^{[x]}}{\sin(x)^3}+\frac{1}{\sin(x)^2}=\lim_{x\to0^-}\frac{1}{\sin(x)^2}\left(\frac{(-1)^{[x]}}{\sin(x)}+1\right)=\lim_{x\to0^-}\frac{1}{\sin(x)^2}\left(\frac{1}{\sin(x)}+1\right)=-\infty$$
because $\frac{1}{\sin(x)}\leq0$ for $x\in]-1,0[$. So the limit does not exist, as left limit and right limit do not agree.

12. Consider the function
$$f(x)=\frac{x(x-1)\tan(x-1)}{x^3-3x+2},$$
whose domain is $\mathbb{R}\setminus\{-2,1\}$.

(a) Study its continuity at $x_0=0$.

(b) Find, if it exists, a continuous extension of the function f in $x_0=1$, or otherwise show that f cannot have a continuous extension at $x_0=1$.

Solution:

(a) Yes, the function is continuous at x_0. We will use the theorems about composition, product, ratio, etc., of continuous functions. The function $x-1$ is continuous everywhere, as it is a polynomial. Thus, since the function tan is continuous everywhere, then so is the composition $\tan(x-1)$. Now, $x(x-1)$ is continuous everywhere, as it is a polynomial; then, as the product of two continuous functions is continuous, then $x(x-1)\tan(x-1)$ is continuous everywhere.

Now, $x^3 - 3x + 2$ is continuous everywhere, as it is a polynomial. To conclude, we will use the fact that the ratio of two continuous functions is continuous, as long as the denominator is not 0. Thus, it suffices to check that $x^3 - 3x + 2$ does not vanish at 0; but this is the case, as its value at $x_0 = 0$ is 2.

(b) If the limit $\lim_{x \to x_0} f(x)$ exists in $\mathbb{R}$ then we can choose it as value a to extend f at x_0. We compute the limit of f at $x_0 = 1$. We can write the denominator of f as $x^3 - 3x + 2 = (x-1)^2(x+2)$ so we get

$$\begin{aligned}
\lim_{x \to 1} f(x) &= \lim_{x \to 1} \frac{x(x-1)\tan(x-1)}{x^3 - 3x + 2} = \lim_{x \to 1} \frac{x(x-1)\tan(x-1)}{(x-1)^2(x+2)} \\
&= \lim_{x \to 1} \left(\frac{x}{x+2} \cdot \frac{\tan(x-1)}{x-1} \right) = \lim_{x \to 1} \frac{x}{x+2} \cdot \lim_{x \to 1} \frac{\tan(x-1)}{x-1} \\
&= \frac{1}{3} \cdot \lim_{x \to 1} \left(\frac{\sin(x-1)}{(x-1)} \cdot \frac{1}{\cos(x-1)} \right) = \frac{1}{3} \cdot 1 = \frac{1}{3}
\end{aligned}$$

(*Attention:* The decomposition of the product of the two limits in the second line exists because both limits exist.)

So the continuous extension of f is

$$\hat{f}_1 : \mathbb{R} \setminus \{-2\} \longrightarrow \mathbb{R}, \qquad \hat{f}_1(x) = \begin{cases} \dfrac{x(x-1)\tan(x-1)}{x^3 - 3x + 2}, & x \neq 1, -2 \\ \frac{1}{3}\,, & x = 1. \end{cases}$$

13. (a) Prove or disprove that a function is continuous if and only if it is uniformly continuous.
    (b) Prove or disprove that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ is a uniformly continuous function.
    (c) Show that the function $f :]0, b[\to \mathbb{R}$ defined by $f(x) = x^2$ is continuous and also uniformly continuous for $b < +\infty$. Show that f is not uniformly continuous when $b = +\infty$.

Solution:

(a) This is not true. The function $f :]0, +\infty[\mapsto \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous, but not uniformly continuous.

(b) Let $\varepsilon > 0$. For $x, y \in \mathbb{R}$, we have

$$|\sin(x) - \sin(y)| = 2 \left| \cos\left(\frac{x+y}{2}\right) \right| \cdot \left| \sin\left(\frac{x-y}{2}\right) \right| \leq 2 \left| \sin\left(\frac{x-y}{2}\right) \right| \leq 2 \left| \frac{x-y}{2} \right| = |x-y|.$$

So if $|x - y| \leq \delta$ with $\delta = \varepsilon$, then $|\sin(x) - \sin(y)| \leq \varepsilon$. Thus $\sin(.)$ is uniformly continuous.

(c) It is continuous because,

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x + x_0||x - x_0| < 2b|x - x_0|$$

So it is enough to take $\delta = \frac{\epsilon}{2b}$.
It is uniformly continuous because,

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < 2b|x - y|$$

So it is enough to take $\delta = \frac{\epsilon}{2b}$.

We want to show that if the domain of f is $(0, \infty)$, then f is not uniformly continuous. It means that we have to find an $\varepsilon > 0$ such that for any $\delta > 0$ we can find $x_0, y_0 \in (0, \infty)$ that satisfy $|x_0 - y_0| < \delta$ but $|f(x_0) - f(y_0)| > \varepsilon$. Let's take $\varepsilon = 1$, $x_0 = N$ some arbitrary integer and $y_0 = N + \delta/2$ for some arbitrary $\delta > 0$. Then

$$|f(x_0) - f(y_0)| = |x_0^2 - y_0^2| = |x_0 - y_0||x_0 + y_0| = |\frac{\delta}{2}||2N + \delta/2|$$

Now if f is uniformly continuous then we require $|f(x_0) - f(y_0)| < 1$ and consequently $|\frac{\delta}{2}||2N + \delta/2| < 1$. But this is a contradiction because for a fixed δ we can pick N to be arbitrary large so that the inequality is violated. So f cannot be uniformly continuous.

14. Let I be an interval, $f\colon I \to \mathbb{R}$ be a continuous function and $f(I)$ the image of I by f. Say if the following statement are true or false.

(a) $f(I)$ is an interval (where here we also admit the degenerate case $f(I) = [m, m] = \{m\}$).

(b) If I is a bounded and closed interval, then $f(I)$ is a bounded and closed interval (where here we also admit the degenerate case $f(I) = [m, m] = \{m\}$).

(c) If I is open, then $f(I)$ is an open interval.

(d) If $I = [a, b[$ with $a, b \in \mathbb{R}$, $a < b$, then f attains its maximum and minimum in I. That is, there exists $m, M \in R(f)$ such that $R(f) = [m, M]$.

Solution:

(a) True. It is a consequence of the intermediate value theorem.

(b) True. It is a consequence of the intermediate value theorem.

(c) False. Take, for example, the function $f\colon]-1, 1[\to \mathbb{R}$ defined by $f(x) = \frac{1}{-x^2+1}$. Then I is open but $f(I) = [1, +\infty[$ is not open.

(d) False. For example take the function $f\colon [-1, 0[\to \mathbb{R}$ defined by $f(x) = \frac{1}{x}\sin\big(\frac{1}{x}\big)$. Then f neither have a minimum nor a maximum on I because for all $n \in \mathbb{N}^*$, we have $-\big(2\pi n \pm \frac{\pi}{2}\big)^{-1} \in I$ but

$$\begin{aligned} f\Big(-\frac{1}{2\pi n + \frac{\pi}{2}}\Big) &= -\Big(2\pi n + \frac{\pi}{2}\Big)\sin\Big(-\Big(2\pi n + \frac{\pi}{2}\Big)\Big) = \Big(2\pi n + \frac{\pi}{2}\Big)\sin\Big(2\pi n + \frac{\pi}{2}\Big) \\ &= \Big(2\pi n + \frac{\pi}{2}\Big)\sin\Big(\frac{\pi}{2}\Big) = 2\pi n + \frac{\pi}{2} > n \end{aligned}$$

and

$$\begin{aligned} f\Big(-\frac{1}{2\pi n - \frac{\pi}{2}}\Big) &= -\Big(2\pi n - \frac{\pi}{2}\Big)\sin\Big(-\Big(2\pi n - \frac{\pi}{2}\Big)\Big) = \Big(2\pi n - \frac{\pi}{2}\Big)\sin\Big(2\pi n - \frac{\pi}{2}\Big) \\ &= \Big(2\pi n - \frac{\pi}{2}\Big)\sin\Big(-\frac{\pi}{2}\Big) = -2\pi n + \frac{\pi}{2} < -n. \end{aligned}$$

15. Find, if it exists, continuous extension of the function $f\colon]0, 1] \to \mathbb{R}$ given by $f(x) = \frac{\tan(\sqrt{1+x}-1)}{x^{3/2}}$ at $x_0 = 0$, or otherwise show that f cannot have a continuous extension at x_0. (*Note: you have to care just about the limit from the right, that is:* $x \to 0^+$)

Solution:
We check if the limit of f exists as $x \to 0^+$. We have

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\tan(\sqrt{1+x}-1)}{x^{3/2}} &= \lim_{x\to 0^+} \frac{\frac{\sin(\sqrt{1+x}-1)}{\cos(\sqrt{1+x}-1)}}{x^{3/2}} \\
&= \lim_{x\to 0^+} \frac{\sin(\sqrt{1+x}-1)}{x^{3/2}} \cdot \frac{1}{\cos(\sqrt{1+x}-1)} \\
&= \lim_{x\to 0^+} \frac{\sin(\sqrt{1+x}-1)}{x^{3/2}} \cdot \frac{\sqrt{1+x}-1}{\sqrt{1+x}-1} \cdot \frac{1}{\cos(\sqrt{1+x}-1)} \\
&= \lim_{x\to 0^+} \frac{\sqrt{1+x}-1}{x^{3/2}} \cdot \frac{\sin(\sqrt{1+x}-1)}{\sqrt{1+x}-1} \cdot \frac{1}{\cos(\sqrt{1+x}-1)}
\end{aligned}$$

Note that the limit of the second fraction is finite due to the sandwich theorem ($\sqrt{1-y^2} \leq \frac{\sin y}{y} \leq 1$) so $\lim_{x\to 0} \frac{\sin(\sqrt{1+x}-1)}{\sqrt{1+x}-1} = 1$. The limit of the last fraction is also finite as $\lim_{x\to 0} \frac{1}{\cos(\sqrt{1+x}-1)} = 1$. For the limit of the first fraction we have:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sqrt{1+x}-1}{x^{3/2}} &= \lim_{x\to 0^+} \frac{\sqrt{1+x}-1}{x^{3/2}} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \\
&= \lim_{x\to 0^+} \frac{x}{x^{3/2}} \cdot \frac{1}{\sqrt{1+x}+1} \\
&= \lim_{x\to 0^+} \frac{1}{x^{1/2}} \cdot \frac{1}{\sqrt{1+x}+1} = +\infty
\end{aligned}$$

So the function $\frac{\tan(\sqrt{1+x}-1)}{x^{3/2}}$ does not have a limit in $\mathbb{R}$ as $x \to 0^+$, hence we cannot have a continuous extension of this function.

16. Use the intermediate value theorem to show that the following equations have at least one solution in $\mathbb{R}$:

 (a) $e^{x-1} = x + 1$

 (b) $x^2 - \dfrac{1}{x} = 1$

Solution:

(a) To use the intermediate value theorem, we must define a continuous function starting from the given equation. So, let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x-1} - x - 1$. Then f is continuous in $\mathbb{R}$ since it is the combination of continuous functions and since $e = 2.7182\ldots$, we have $f(2) = e - 3 < 0$ and $f(3) = e^2 - 4 > 0$. By the intermediate value theorem, there exist $x_0 \in [2, 3]$ such that $f(x_0) = 0$.

 Note also that this equation also admits another root. In fact, we have $f(0) = \frac{1}{e} - 1 < 0$ and $f(-1) = \frac{1}{e^2} > 0$ and by the intermediate value theorem there exists $x_0 \in [-1, 0]$ such that $f(x_0) = 0$.

(b) To use the intermediate value theorem, we must define a continuous function starting from the given equation. Since the given equation is not defined at $x = 0$, we need to define the function f on $]-\infty, 0[$ and on $]0, \infty[$

If $x < 0$, we have $x^2 - \frac{1}{x} = x^2 + \frac{1}{|x|} > 1$ because one of the two terms is always ≥ 1 so the equation does not take any roots. So we define $f\colon]0, \infty[\to \mathbb{R}$, $f(x) = x^2 - \frac{1}{x} - 1$. This function is continuous (sum of continuous functions) and we have $f(1) = -1 < 0$ and $f(2) > 0$. By the intermediate value theorem, there exists $x_0 \in [1, 2]$ such that $f(x_0) = 0$.

17. State if the following functions are continuous and differentiable at $x = 0$.

(a) $|\sin(x)|$

(b) $|x^3|$

Solution:

(a) The function $\sin(x)$ is continuous and differentiable at $x = 0$. The function $|.|$ is continuous, so $|\sin(x)|$ is continuous, as it is the composition of two continuous functions. If we look at the graph of $|\sin(x)|$ we see that there are two tangent lines at $x = 0$. Hence, we expect that the function is not differentiable at $x = 0$. Let's compute the derivative, if it exists. We distinguish right and left limits

$$\lim_{x \to 0^+} \frac{|\sin(x)|}{x} = \lim_{x \to 0^+} \frac{\sin(x)}{x} = 1, \qquad \lim_{x \to 0^-} \frac{|\sin(x)|}{x} = \lim_{x \to 0^-} \frac{-\sin(x)}{x} = -1.$$

So the limit $\lim_{x \to 0} \frac{|\sin(x)|}{x}$ does not exist and $|\sin(x)|$ is not differentiable at $x = 0$.

(b) The function x^3 is continuous and differentiable everywhere because it is a polynomial. Since $|.|$ is continuous, then $|x^3|$ is continuous everywhere because of composition of continuous functions. The derivative at $x = 0$, if it exists, is the limit $\lim_{x \to 0} \frac{|x^3|}{x}$. We distinguish the right and left limit.

$$\lim_{x \to 0^+} \frac{|x^3|}{x} = \lim_{x \to 0^+} \frac{x^3}{x} = 0, \qquad \lim_{x \to 0^-} \frac{|x^3|}{x} = \lim_{x \to 0^-} \frac{-x^3}{x} = 0.$$

Hence the limit exists and $|x^3|$ is differentiable at $x = 0$.

18. Let f and g be two continuous functions in $[a, b]$, such that $f(a) > g(a)$ and $f(b) < g(b)$. Show that there is $c \in]a, b[$ such that $f(c) = g(c)$. (*Hint: use the function $h = f - g$ and the intermediate value theorem.*)

Solution:

Define the function $h(x) = f(x) - g(x)$. Since f and g are continuous functions in $[a, b]$ then h is also a continuous function on $[a, b]$. Also we have that $h(a) = f(a) - g(a) > 0$ since $f(a) > g(a)$ and $h(b) = f(b) - g(b) < 0$ since $f(b) < g(b)$. So h satisfies the intermidiate value theorem so there exists a $c \in]a, b[$ such that $h(c) = 0$. But $h(c) = 0$ implies that $f(c) - g(c) = 0$ meaning that $f(c) = g(c)$.

19. Find the inverse of the following functions if they exist. Give the domain of both functions.

(a) $f(x) = \sqrt{(2x + 4)^3 - 7}$

(b) $f(x) = \frac{2x+3}{3x+5}$

(c) $f(x) = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}$

Solution:

(a) To find the domain of f we notice that only positive values are accepted under the square root sign:

$$(2x+4)^3 - 7 \geq 0 \Longrightarrow x \geq -2 + \frac{\sqrt[3]{7}}{2}$$

So $D_f = [-2 + \frac{\sqrt[3]{7}}{2}, +\infty[$. We observe that f is injective because it is a composition of injective functions: $\sqrt{x}$, $x - 7$, x^3 and $2x + 4$ are all injective. Hence, an inverse function exists. To find the inverse function we have:

$$y = \sqrt{(2x+4)^3 - 7} \Longrightarrow \frac{\sqrt[3]{y^2+7}}{2} - 2 = x$$

So the inverse function is $f^{-1}(x) = \frac{\sqrt[3]{x^2+7}}{2} - 2$ and its domain is the entire real numbers $D_{f^{-1}} = \mathbb{R}$.

(b) To find the domain of f we notice that the denominator cannot become zero so $D_f = \mathbb{R} \backslash \{-5/3\}$. To find the inverse function we have:

$$y = \frac{2x+3}{3x+5} \Longrightarrow 3yx + 5y = 2x + 3 \Longrightarrow x(3y-2) = 3 - 5y \Longrightarrow x = \frac{3-5y}{3y-2}$$

where we assumed that $x \neq -5/3$ and $y \neq 2/3$. So x is uniquely determined by y, in particular the function f is injective. So the inverse function exists and is given by $f^{-1}(x) = \frac{3-5x}{3x-2}$ with the domain $D_{f^{-1}} = \mathbb{R} \backslash \{2/3\}$.

(c) Note that $2 \sin x \cos x = \sin(2x)$. Since the denominator cannot become zero, we have that $x \neq \frac{k\pi}{2}$ where $k \in \mathbb{Z}$. So the domain of f is $\mathbb{R} \backslash \{\frac{k\pi}{2} | k \in \mathbb{Z}\}$. We have

$$f(x) = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} = \frac{\cos(2x)}{\sin(2x)} = \cot(2x)$$

We observe that the function f is not injective because it is periodic. Hence the inverse function does not exist. But $g := f|_{]0, \frac{\pi}{2}[} :]0, \frac{\pi}{2}[\to \mathbb{R}$ is injective with inverse function $g^{-1}(x) = \frac{1}{2} \operatorname{arccot} x$. The domain of the inverse function is $D_{g^{-1}} = \mathbb{R}$.

20. Study the continuity of the function $f \colon \mathbb{R} \to \mathbb{R}$ in $x = 0$, where

$$f(x) = \begin{cases} \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Solution:

Consider the sequences (x_n) and (y_n) respectively defined as $x_n = \frac{1}{2n\pi}$ et $y_n = \frac{1}{\frac{\pi}{2} + 2n\pi}$. These sequences satisfy $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$ but

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \cos(2n\pi) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \cos\left(\frac{\pi}{2} + 2n\pi\right) = 0.$$

So, $\lim_{x\to 0} f(x)$ does not exist and f is not continuous at $x = 0$.

21. State if the following statements are true or false:

(a) If $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = c \in \mathbb{R}$ then f is continuous at 0.

(b) If $f \circ g$ is continuous, then f is continuous.

(c) For every $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, there exists at most one value $a \in \mathbb{R}$ such that the function $\hat{f} : \mathbb{R} \to \mathbb{R}$

$$\hat{f}(x) := \begin{cases} f(x) & x \neq 0, \\ a & x = 0, \end{cases}$$

is continuous.

(d) If $|f(x)|$ is continuous everywhere then $f(x)$ is also continuous.

(e) If $f \circ g$ is continuous, then g is continuous.

(f) If functions f and g are continuous everywhere then f/g is also continuous everywhere.

(g) If $f(x)$ is continuous everywhere, then $|f(x)|$ is continuous everywhere.

(h) If the composition $f \circ g$ is not continuous at $x = a$, then g is not continuous at $x = a$ or f is not continuous at $g(a)$.

Solution:

(a) False. Take

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

which has the left and right limits 0 at $x = 0$ but $f(0) = 1$. So f does not have a limit.

(b) False. If g is constant, then $f \circ g$ is constant, hence continuous. We can then choose any non-continuous function f to obtain a counterexample.

(c) True. If $\hat{f}$ is continuous, then in particular it is continuous at 0. Then $a = \hat{f}(0) = \lim_{x\to 0} f(x)$ is uniquely determined by f.

(d) False. Take for example

$$f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

(e) False. If f is constant, then $f \circ g$ is constant, hence continuous. We can then choose any non-continuous function g to obtain a counterexample.

(f) False. The function might not be defined at the zeros of g. (Notice that as soon as we are away from the zeroes of g, the ratio of the two continuous functions f/g is continuous.)

(g) True. The composition of two continuous functions is continuous. Here $f(x)$ and $|x|$ are continuous everywhere.

(h) True. To see why, we should look at the continuity of composite functions. For functions f and g and a point $x_0 = a$, if g is continuous at x_0 and f is continuous at $g(x_0)$ then $f \circ g$ is continuous at $x_0 = a$. The contrapositive state is also true: If the composition $f \circ g$ is not continuous at a, then g is not continuous at a or f is not continuous at $g(a)$.