Analysis I (English)
Fall Semester 2025–2026

Analysis 1 - Exercise Set 8

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Using the definition, state if the following functions are injective, surjective or bijective. If the function is bijective, find the inverse function.
   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^5$
   (b) $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$

2. For the two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ below, find $g \circ f$ and $f \circ g$.

$$f(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}, \qquad g(x) = \begin{cases} 2x - 3 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases}$$

3. State if the following are true or false.
   (a) The function $f = \sqrt{1 - \cos x}$ is even.
   (b) There is no function which is both even and odd.
   (c) Let f be an odd function. If f is bijective, then f^{-1} is also odd.

4. Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, Determine the monotonicity (increasing or decreasing) of the composition $g \circ f : \mathbb{R} \to \mathbb{R}$ in the following cases:
   (a) if f and g are both increasing.
   (b) if f and g are both decreasing.
   (c) if f is increasing and g is decreasing. What can we say about $f \circ g$?

5. Using the definition, state if the following functions are injective, surjective or bijective. If the function is bijective, find the inverse function.
   (a) $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin x$
   (b) $f : [0, \pi] \to [-1, 1]$, $f(x) = \cos x$

6. For the two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ below, find $g \circ f$ and $f \circ g$.

$$f(x) = \begin{cases} |2x - 1| & \text{if } x \geq -1 \\ -x(x + 2) & \text{if } x < -1 \end{cases}, \qquad g(x) = \begin{cases} -\sqrt{x - 4} & \text{if } x \geq 4 \\ 1 - x/2 & \text{if } x < 4 \end{cases}$$

7. State if the following are true or false.
   (a) If f is an even function and g is an odd function, then $h = f \cdot g$ is an odd function.
   (b) If f is an even function and g is an odd function, then $h = f \circ g$ is an odd function.
   (c) A function is either even or odd or both.

8. Calculate the following limits.

(a) $\lim_{x\to 0} \frac{x^3+4x}{2x}$

(b) $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$

(c) $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$

9. Calculate the following limits.

(a) $\lim_{x\to 4} \frac{x^2+5x-36}{x^2-16}$

(b) $\lim_{x\to 1} \frac{x^n-1}{x-1}$ (*Hint: Try to factorize $x-1$ from the numerator.*)

10. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{500}}{(1.0001)^n}$$

(a) converges absolutely.

(b) converges, but not absolutely.

(c) approaches $+\infty$.

(d) approaches $-\infty$.

11. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sqrt{n}+1} - \frac{n+1}{\sqrt{n+1}+1} \right)$$

(a) diverges.

(b) converges to $\frac{1}{2} - \frac{2}{\sqrt{2}+1}$.

(c) converges to $\frac{1}{2}$.

(d) converges to 0.

12. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$$

is

(a) divergent.

(b) converges to e.

(c) converges to e^{-1}.

(d) converges to 1.

13. (Multiple choice) The limit

$$\lim_{n\to\infty} \frac{\sqrt[3]{1-\frac{1}{n}}-1}{\sqrt[4]{1-\frac{1}{n}}-1}$$

is

(a) $\frac{3}{4}$

(b) $\frac{4}{3}$

(c) ∞

(d) 0

14. (Multiple choice) The limit

$$\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n \frac{n+2}{n+1}$$

is

(a) e^2

(b) e

(c) ∞

(d) 0

15. (Multiple choice) The limit

$$\lim_{n\to\infty} n^2 \cdot \sin\left(\frac{2n+3}{n^3}\right)$$

is

(a) 0

(b) $\frac{1}{2}$

(c) 2

(d) ∞

16. Find the values $\alpha \in \mathbb{R}$ such that the limit $\lim_{x\to\alpha} \dfrac{\tan(x-\alpha)^2}{(x-\alpha)^2}$ exists in $\mathbb{R}$.

17. Compute the following limits if they exist.

(a) $\lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{1-x^3}\right)$

(b) $\lim_{x\to a} \frac{\cos(x)-\cos(a)}{x-a}$ with $a \in \mathbb{R}$

(c) $\lim_{x\to+\infty}(x|\sin(x)| - x^2 + 4)$

18. Find the values $\alpha, \beta \in \mathbb{R}$ such that the limit $\lim_{x\to 0} \dfrac{x^2 \sin\left(\frac{1}{x}\right) + \alpha|x|}{\sqrt{x^2 + \beta\left|\cos\left(\frac{1}{x}\right)\right|}}$ exists in $\mathbb{R}$.