

Analysis 1 - Exercise Set 7

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. State if the following statements are true or false. If you think the statement is true, then prove that; otherwise, provide a counterexample.

- If (x_n) is a sequence that converges to 0, then the series $\sum_{n=0}^{\infty} x_n$ converges.
- Let (x_n) and (y_n) be two sequences such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=0}^{\infty} x_n$ diverges, then the series $\sum_{n=0}^{\infty} y_n$ diverges.
- Let (x_n) and (y_n) be sequences such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=0}^{\infty} x_n$ diverges, then the series $\sum_{n=0}^{\infty} y_n$ diverges.
- Let (x_n) and (y_n) be sequences. If the series $\sum_{n=0}^{\infty} x_n$ converges and the sequence (y_n) converges, then the series $\sum_{n=0}^{\infty} x_n y_n$ converges.

2. For each of the following, determine whether the series is convergent or divergent.

- $\sum_{n=0}^{\infty} \frac{1}{n^2+n+3}$
- $\sum_{n=0}^{\infty} \frac{2n^2+1}{3n^2+2}$
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

3. (a) Show that for every $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k=1}^n \frac{(-1)^{k+1}k}{k^2 - \frac{1}{4}} = 1 + \frac{(-1)^{n+1}}{2n+1}.$$

- Use the previous part to compute $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{k^2 - \frac{1}{4}}$.
- Is the series absolutely convergent?

4. For each of the following, determine whether the series is convergent or divergent.

- $\sum_{n=0}^{\infty} \frac{\sin(2n^2)}{n^2+3}$
- $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$
- $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n+3}$

5. Let $a \in \mathbb{R}$ such that $a > 0$, $a \neq 1$. Determine for which values of a the series

$$\sum_{n=1}^{\infty} \frac{a^n}{a^{2n} - 1}$$

converges. (*Hint: use the Cauchy criterion.*)

6. For each of the following, determine whether the series is convergent or divergent.

- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+3)}}$

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^5}}{n^3+1}$

(c) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$

7. State if the following statements are true or false. If you think the statement is true, then prove that; otherwise, provide a counterexample.

(a) Let (a_n) be any sequence and $x_0 \in \mathbb{R}$. Then there exists at least one element $x \in \mathbb{R}$ such that the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges.

(b) Let (x_n) be a sequence. If $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| \leq 1$, then the series $\sum_{n=0}^{\infty} x_n$ converges.

8. (a) Find $A, B \in \mathbb{R}$ such that for every $k > 0$ we have

$$\frac{1}{k(k+4)} = \frac{A}{k} + \frac{B}{k+4}.$$

(b) Use the previous part to compute $\sum_{k=1}^n \frac{1}{k(k+4)}$ for all $n \geq 5$.

(c) Use the previous part to compute $\sum_{k=1}^{\infty} \frac{1}{k(k+4)}$.

9. (Multiple choice) The series

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^n$$

is

(a) divergent.

(b) converges to $2 + \sqrt{2}$.

(c) converges to $2 - \sqrt{2}$.

(d) cannot be determined.

10. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{500}}{(1.0001)^n}$$

(a) converges absolutely.

(b) converges, but not absolutely.

(c) approaches $+\infty$.

(d) approaches $-\infty$.

11. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sqrt{n+1}} - \frac{n+1}{\sqrt{n+1}+1} \right)$$

(a) diverges.

(b) converges to $\frac{1}{2} - \frac{2}{\sqrt{2}+1}$.

(c) converges to $\frac{1}{2}$.

(d) converges to 0.

12. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

- (a) converges absolutely.
- (b) converges, but not absolutely.
- (c) diverges to $+\infty$.
- (d) diverges to $-\infty$.

13. (Multiple choice) The series

$$\sum_{n=2}^{\infty} \frac{n^{1/3}}{n^{4/3} - 1}$$

is

- (a) convergent.
- (b) absolutely convergent.
- (c) diverges.
- (d) cannot be determined.

14. Terminate the proof that we started in class showing the convergence of $\sum_{i=1}^{\infty} \frac{(-1)^i}{i}$. This is what we have proven in class and that you can assume:

- (a) the subsequence (y_k) of (s_n) ,

$$y_k := s_{2k+1} = \sum_{i=0}^{2k+1} \frac{(-1)^i}{i}$$

is strictly increasing;

- (b) (y_k) is bounded; in particular (y_k) converges to a limit $y \in \mathbb{R}$.

(Hint: Show that (s_n) is a Cauchy sequence. Use the fact that since (y_k) converges, then it is Cauchy, and that $s_{2k} - \frac{1}{2k+1} = s_{2k+1}$.)

15. (a) Show that for every $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k=1}^n (-1)^{k+1} \frac{k}{k^2 - \frac{1}{4}} = 1 + \frac{(-1)^{n+1}}{2n+1}.$$

- (b) Use the previous part to compute $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 - \frac{1}{4}}$.

- (c) Is the series absolutely convergent?

16. Compute $\lim_{n \rightarrow \infty} \sqrt[n]{n!}$

17. Let (x_n) be a sequence.

- (a) Show that if $\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = \rho$, with $\rho > 1$, then (x_n) is unbounded. In particular, it diverges.
- (b) Show that if $\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = \rho$, with $0 \leq \rho < 1$, then (x_n) converges to 0. In particular, it is bounded.
- (c) Provide two sequences (y_n) and (z_n) with the following properties: (y_n) converges and $\lim_{n \rightarrow \infty} \sqrt[n]{|y_n|} = 1$, and (z_n) diverges to $+\infty$ and $\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = 1$

18. Let $(t_n) \subset \mathbb{R}^*$ be a sequence. Assume that $\lim_{n \rightarrow \infty} t_n = 0$. Show that $\lim_{n \rightarrow \infty} \frac{\sin(t_n)}{t_n} = 1$.
(Hint: recall that for $x \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} 0 \leq \sin(x) \leq x \leq \tan(x) &\Rightarrow 1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)} \Rightarrow \cos(x) \leq \frac{\sin(x)}{x} \leq 1 \\ &\Rightarrow \cos(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \Rightarrow 1 - \sin(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \\ &\Rightarrow 1 - x^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \Rightarrow \sqrt{1 - x^2} \leq \frac{\sin(x)}{x} \leq 1. \end{aligned}$$

19. Show that if $\lim_{n \rightarrow \infty} \frac{|x_{n+1}|}{|x_n|} = 1$, then anything can happen for $\sum_{n=0}^{\infty} x_n$. That is, it is possible to find sequences (x_n) such that:

- (a) (x_n) is unbounded;
- (b) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ diverges;
- (c) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ converges absolutely;
- (d) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ converges but not absolutely.

For each item above, provide an example.

20. For each of the following, determine whether the series is convergent or divergent.

(a) $\sum_{k=1}^{\infty} \frac{\pi^k}{k \cdot 2^k}$

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+4} - \sqrt{n+2}}{n}$

(c) $\sum_{k=2}^{\infty} \frac{k^2 - 1}{(k-1)^3}$