Analysis 1 - Exercise Set 6

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Compute, if they exists, the limits of the following sequences

   (a) $\sqrt[n]{\frac{3}{n}}$

   (b) $(-1)^n \left(\frac{n^2+1}{n-1}\right)$

   (c) $\frac{1}{n^2}\left(\sqrt{1+n+\pi n^2+\frac{\sin(n)}{n}}-1\right)$

   (d) $\sqrt[n]{n\log(n)}$ (*Hint*: $1 < \log(n) < n$ for $n > 3$)

   (e) $n^2\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1\right)$

   (f) $\left(\frac{n-1}{n}\right)^{n^2}$

   (g) $\sqrt[n]{\frac{2n}{3n^2-1}}$

   (h) $\frac{4n^2-2\pi}{-n^3+\sqrt{7}n}$

   (i) $\frac{(n+1)!}{n!-(n+1)!}$

   (j) $\frac{\sqrt{\frac{cos(n)}{n^2}+1}-1}{\sqrt{e-\frac{1}{n}}-\sqrt{e}}$

2. Let $a, b \in \mathbb{R}_+$ and (x_n) be a sequence defined by the recurrence relation

$$x_{n+1} = ax_n^2 \quad x_0 = b.$$

   (a) Show by induction that every element in the sequence (x_n) is given by

$$x_n = a^{2^n-1}b^{2^n}.$$

   (b) Use part (a) to compute

$$\lim_{n\to+\infty} x_n.$$

3. Show that the following recursive sequence is convergent and calculate the limit

$$a_n = \frac{7}{3} - \frac{1}{1+a_{n-1}}, \quad a_1 = 1.$$

4. This question is going to show that, whenever we have a sequence that is defined recursively, we need to show that it converges, and that computing the candidates for the limit is not enough.

   Consider the sequence defined as $a_1 = 10$, $a_{n+1} = a_n^2$ for $n \geq 1$.

(a) Show that, if the limit of (a_n) exists, then it is either 0 or 1.

(b) Show that (a_n) diverges to $+\infty$.

5. Compute the limit of $a_n = \left(\frac{n+3}{n+1}\right)^n$ using subsequences. (*Hint: first, manipulate the definition of* a_n *so that it looks more to the sequence of a previous exercise, then use the subsequence with odd indices.*)

6. State if the following statements are true or false. If you think the statement is true, then prove that; otherwise, provide a counterexample.

   (a) If a sequence is not bounded above, it must be increasing.

   (b) Any monotone sequence has a convergent subsequence.

   (c) If (a_n) has no divergent subsequence, then (a_n) is convergent.

   (d) If (a_n) is Cauchy convergent, then also $(|a_n|)$ is Cauchy convergent.

   (e) If (a_n) is a Cauchy sequence, then the sequence $b_n = c \cdot a_n$, $c \neq 0$ is a Cauchy sequence.

   (f) If (a_n) is Cauchy, there exists $\varepsilon > 0$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \in \mathbb{N}$.

   (g) Any sequence has a convergent subsequence.

   (h) If (a_n) and (b_n) are Cauchy sequences, then the sequence $c_n = a_n + b_n$ is a Cauchy sequence.

7. Show if the sequence

$$a_n = \frac{\sin(a_{n-1}) + 1}{2} \qquad a_1 = 0$$

satisfies the definition of Cauchy sequence. (*Hint: Use the trigonometric formulas from Exercise Sheet 1*)

8. Let (a_n) and (b_n) be two sequences. Show the following facts.

   (a) Assume that (a_n) and (b_n) are bounded. Prove that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

   (b) Provide an example of sequences (a_n) and (b_n) such that the inequality in part (a) is strict.

   (c) Assume that $\liminf a_n = 5$. Show that there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, $a_n \geq 4$.

   (d) Assume (b_n) is defined as follows:

$$b_n = \begin{cases} \frac{100}{n} \text{ if } 3|n \\ 2 - \frac{1}{n} \text{ if } 3|n-1 \\ \frac{1}{2} \text{ if } 3|n-2 \end{cases}$$

   Compute $\limsup b_n$, $\liminf b_n$, and exhibit a subsequence of (b_n) converging to $\limsup b_n$ and a subsequence converging to $\liminf b_n$.

9. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

   (a) If

$$\lim_{n\to\infty} a_n = 0,$$

   then

$$\lim_{n\to\infty} (a_n \sin(n)) = 0.$$

(b) If (a_n) is bounded, then
$$\lim_{n\to\infty} (a_n e^{-n}) = 0.$$

(c) If
$$\lim_{n\to\infty} a_n = 0,$$
then the sequence $b_n := a_n e^n$ is unbounded.

10. Compute the following limits:

(a) $\lim\limits_{n\to\infty} \frac{2^n-3^n}{3^n+1}$

(b) $\lim\limits_{n\to\infty} n^3 \left(1-\cos(\frac{1}{n})\right)\sin(\frac{1}{n})$
(*Hint: Use the fact that* $\lim\limits_{m\to\infty} \frac{\sin(\frac{1}{m})}{\frac{1}{m}} = 1$ *and* $\lim\limits_{m\to\infty} \cos\left(\frac{1}{m}\right) = 1$.)

(c) $\lim\limits_{n\to\infty} \frac{\sin^2(n)}{2^n}$

(d) $\lim\limits_{n\to\infty} n(\sqrt{n^4+6n+3}-n^2)$

11. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

(a) If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$
then (a_n) converges.

(b) If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$
then (a_n) diverges.

12. Determine if the sequence (a_n) is convergent or not in the following cases.

1. $a_n = \frac{n}{e^n}$.
2. $a_n = \frac{10^n}{n!}$
3. $a_n = \frac{n^n}{e^n}$
4. $a_n = \frac{n!}{n^n e^{\frac{n}{2}}}$

13. Determine if the following sequences converge or not. If the sequence is convergent, determine its limit.

(a) $a_n = \frac{3n^2-1}{10n+5n^2}$

(b) $a_n = \frac{3^{2n}}{n}$

(c) $a_n = \frac{(-1)^n n^2}{2^n}$

14. Compute the following limits:

(a) $\lim\limits_{n\to\infty} \frac{n^3}{7^n}\cos\left(n^2\right)$

(b) $\lim\limits_{n\to\infty} \dfrac{\sin(n+1)-\sin(n-1)}{\cos(n+1)+\cos(n-1)}$
(*Hint: Use trigonometric formulas from Exercise Sheet 1*)

(c) $\lim_{n\to\infty} \frac{\sin\left(\sqrt{n^3+n^2+1}\right)}{n^3+n^2+1}$

15. Give an example of a sequence (x_n) such that the sequence $y_n = x_{n+1} - x_n$ converges to 0 but (x_n) itself is divergent.

16. Prove that if $\lim_{n\to\infty} x_n = +\infty$ and (y_n) bounded from below, then $\lim_{n\to\infty}(x_n + y_n) = +\infty$. Show that this is true also if $+\infty$ is replaced with $-\infty$ and (y_n) is assumed to be bounded from above.

17. Prove that if $x_n \neq 0$, for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} |\frac{x_n}{x_{n-1}}| = +\infty$ then (x_n) is unbounded and, thus it diverges. Construct examples of sequences (x_n) satisfying the conditions above and such that $\lim_{n\to\infty} x_n = +\infty$ (resp. $\lim_{n\to\infty} x_n = -\infty$).

18. Consider the recursive sequence $a_{n+1} = 7 - \frac{10}{a_n}$, with initial datum $a_1 = 4$. Compute the first three values. Then, show that it is bounded by 2 and 5, that it is increasing, and then compute the limit.

19. Consider the recursive sequence $a_{n+1} = \sqrt{8a_n - 7}$, with initial datum $a_1 = 4$. Show that it is bounded by 1 and 7, that it is increasing, and then compute the limit.

20. Find the limit for $x_n = \frac{\sin(x_{n-1})}{2}$, $x_0 = 1$. [*Hint:* use the fact $|\sin(x)| \leq |x|$ for all x]

21. Let $(a_n), (b_n)$ be sequences. State if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

    (a) If (a_n) is monotone, then $\lim_{n\to\infty} a_n$ exists or $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$.

    (b) If (a_n) and (b_n) are monotone, then the sequence $c_n = a_n + b_n$ is monotone.

    (c) If $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$, then (a_n) is a bounded sequence.

    (d) An unbounded sequence can have a convergent subsequence.

    (e) If (a_n) has no convergent subsequence, then (a_n) is unbounded.