Analysis 1 - Exercise Set 10

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Let I be an interval, $f\colon I \to \mathbb{R}$ be a continuous function and $f(I)$ the image of I by f. Say if the following statement are true or false.

   (a) If I is bounded, then $f(I)$ is bounded.

   (b) If $I = [a, \infty[$ with $a \in \mathbb{R}$, then f attains its maximum and minimum in I.

   (c) If f is strictly increasing and I is open, then $f(I)$ is open.

2. Find, if it exists, continuous extension of the function $f :]2, \infty[\to \mathbb{R}$ give by $f(x) = \frac{\sqrt{x}-\sqrt{2}+\sqrt{x-2}}{\sqrt{x^2-4}}$ at $x_0 = 2$, or otherwise show that f cannot have a continuous extension at x_0.

3. **The Bisection Algorithm:** Using the intermediate value theorem and successive bisection of the interval $[0, 1]$, find an interval of the length $L \leq \frac{1}{8}$ that contains a solution of the equation
$$x^3 + x - 1 = 0.$$

4. Let the function $f\colon [0, \infty) \to \mathbb{R}$ be defined as
$$f(x) = \begin{cases} \dfrac{3x^2 - 10x + 3}{x^2 - 2x - 3}\,, & x > 3 \\ \alpha\,, & x = 3 \\ \beta x - 4\,, & x < 3 \end{cases}$$
   Find $\alpha, \beta \in \mathbb{R}$ such that the function is continuous at $x = 3$.

5. Show that if $f(x)$ is continuous on $[-1, 1]$ and $f(-1) = f(1)$, then there exists $\delta \in [0, 1]$ such that $f(\delta) = f(\delta - 1)$.

6. Find, if it exists, continuous extension of the function $f : [-\pi/4, 0[\cup]0, \pi/4] \to \mathbb{R}$ given by $f(x) = \frac{1-\cos x}{\tan^2 x}$ at $x_0 = 0$, or otherwise show that f cannot have a continuous extension at x_0.

7. Let us define the functions
$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

   (a) Find domain and range for each of the 3 functions.

   (b) Show that
$$\cosh(x)^2 - \sinh(x)^2 = 1.$$

   (c) Find a suitable domain, for each of the 3 functions, over which the function is invertible.

   (d) Compute
$$\lim_{x\to+\infty} \cosh(x), \quad \lim_{x\to-\infty} \cosh(x),$$
$$\lim_{x\to+\infty} \sinh(x), \quad \lim_{x\to-\infty} \sinh(x),$$
$$\lim_{x\to+\infty} \tanh(x), \quad \lim_{x\to-\infty} \tanh(x).$$

8. Calculate the derivative f' of the function f and give the domain of f and f'.

   (a) $f(x) = \dfrac{5x+2}{3x^2-1}$
   (b) $f(x) = \tan(x)$
   (c) $f(x) = x\sin(x) + \frac{\cos(x)^2}{x^2+2}$

9. Prove the quotient rule for derivatives:
   if $f : I \to \mathbb{R}$, $f(x) = \frac{g(x)}{h(x)}$, $x_0 \in I$ and both g and h are differentiable at x_0, with $h(x_0) \neq 0$, then, $f'(x_0) = \frac{g'(x_0)h(x_0)-g(x_0)h'(x_0)}{h(x_0)^2}$.

10. For each of the following functions, find the inverse function and the derivative of the inverse function.

   (a) $f(x) = \cos x$, $x \in \,]0, \pi[$.
   (b) $f(x) = \tan x$, $x \in \,]-\frac{\pi}{2}, \frac{\pi}{2}[$.

11. Calculate the derivative f' of the function f and give the domain of f and f'.

   (a) $f(x) = \dfrac{x^2}{\sqrt{1-x^2}}$
   (b) $f(x) = \sin(x)^2 \cdot \cos(x^2)$
   (c) $f(x) = \sqrt{\sin\left(\sqrt{\sin(x)}\right)}$
   (d) $f(x) = \sin(x)\log(\sin(x))e^{\cos(x)}$

12. For $x \in \mathbb{R}$, e^x has been defined as $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$. Hence, this definition gives rise to a function $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$. Prove the following properties of the exponential e^x:

   (a) $e^0 = 1$;
   (b) $e^x \cdot e^y = e^{x+y}$; [For this part of the exercise you can assume the following result:
   Let $(a_n), (b_n)$ be sequences. Assume that both $\sum\limits_{i=0}^{\infty} a_i, \sum\limits_{i=0}^{\infty} b_i$ converge to a finite limit, and, moreover, that at least one of $\sum\limits_{i=0}^{\infty} a_i, \sum\limits_{i=0}^{\infty} b_i$ converges absolutely. Then the sequence (z_n), $z_n := \sum_{l=0}^{n} a_l b_{n-l}$ satisfies
   $$\sum_{i=0}^{\infty} a_i \cdot \sum_{i=0}^{\infty} b_i = \sum_{i=0}^{\infty} z_i.$$
   (c) $e^{-x} = \frac{1}{e^x}$
   (d) e^x is a strictly increasing function of x; e^{-x} is a strictly decreasing function of x;
   (e) Use the definition of $\log(x)$ as inverse of the function e^x to show that
      (i) $\log(ab) = \log(a) + \log(b)$ for all $a, b > 0$.
      (ii) $\log(a^b) = b\log(a)$ for all $a > 0$ and all $b \in \mathbb{R}$.
      (iii) $\log(x)$ is a strictly increasing function of x.

13. For each function, calculate $f^{(n)}$, the n-th order derivative of f.

   (a) $f(x) = x^m \quad (m \in \mathbb{Z})$
   (b) $f(x) = \sin(2x) + 2\cos(x)$

(c) $f(x) = \log(x)$

14. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. State if the following are true or false.

    (a) f even $\quad\Rightarrow\quad f'$ odd,
    (b) f odd $\quad\Rightarrow\quad f'$ even,
    (c) f' even $\quad\Rightarrow\quad f$ odd,
    (d) f periodic $\quad\Rightarrow\quad f'$ periodic.

15. Calculate $(g \circ f)'(0)$ for the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ defined by

    (a) $f(x) = 2x + 3 + (e^x - 1)\sin(x)^7\cos(x)^4$ and $g(x) = \log(x)^3$.

    (b) $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) + 2x\,, & x \neq 0 \\ 0\,, & x = 0 \end{cases}$ and $g(x) = (x-1)^4$.

16. Calculate the derivative f' of the function f and give the domain of f and f'.

    (a) $f(x) = \sqrt[5]{\left(2x^4 + e^{-(4x+3)}\right)^3}$
    (b) $f(x) = e^{\sqrt[3]{\log(4x)^2}}$
    (c) $f(x) = \log\left(4^{\sin(x)}\right)e^{\cos(4x)}$

17. State if the following are true or false.

    (a) If f is differentiable at $a \in \mathbb{R}$, Then there is $\delta > 0$ such that f is continuous on $]a-\delta, a+\delta[$.
    (b) If f is differentiable from left and right at $a \in \mathbb{R}$, then f is differentiable at a.
    (c) If f is differentiable on $\mathbb{R}$, then $g(x) = \sqrt{f^2(x)}$ is differentiable on $\mathbb{R}$.

18. For each of the following functions, find the inverse function. Find the derivative of the inverse function once by direct calculation and once by the inverse function derivative.

    (a) $f(x) = \sqrt{x^2 + 9}$, $x \geq 0$.
    (b) $f(x) = \frac{1}{1+x}$, $x \neq -1$.

19. Find maximum and minimum of the following functions

    (a) $f(x) = x^2 - 5$ in $[-\pi, \pi]$
    (b) $f(x) = \sqrt[3]{(x-1)(x-2)^2}$ in $[1 + \frac{1}{10}, 2 - \frac{1}{10}]$

20. Calculate f'

    (a) $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{\cos x}{2+\sin(\log x)}$
    (b) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \log(a|x|)$, $a > 0$
    (c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x^2 \sin x}$