Analysis 1 - Exercise Set 9

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Show that:
   (a) $\lim_{x\to+\infty} e^x = +\infty$;
   (b) $\lim_{x\to+\infty} \arctan(x) = \frac{\pi}{2}$, $\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$;
   (c) $\lim_{x\to+\infty} \arctan(e^x) = +\frac{\pi}{2}$.

2. Make an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous only at one point. Can you generalize this to make an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous only at n points for fixed $n \in \mathbb{N}$?

3. Use the ϵ, δ definition of continuity to show that $f(x) := \sin(x)$ is continuous everywhere. (Hint: Using geometry, show $\sin(\theta) \leq \theta \quad \forall\theta \in [0, \frac{\pi}{2}]$. Then, using this result show that $|\sin(\theta)| \leq |\theta| \quad \forall\theta \in \mathbb{R}$.)

4. We say that a function f defined on a pointed neighborhood E of $x_0 \in \mathbb{R}$ has a continuous extension at the point x_0 if there exists a number $a \in \mathbb{R}$ such that the function

$$\hat{f}(x) := \begin{cases} f(x) & \text{if } x \in E, \\ a & \text{if } x = x_0, \end{cases}$$

   is continuous at x_0.

   Find, if it exists, a continuous extension of the function $f\colon [0, 1[\cup]1, \infty[\to \mathbb{R}$,

$$f(x) = \frac{\sqrt{x+1} - \sqrt{2x}}{\sqrt{1+2x} - \sqrt{3}}$$

   in $x_0 = 1$, or otherwise show that f cannot have a continuous extension at x_0. Finally, compute $\lim_{x\to+\infty} f(x)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} \sin(x) & \text{if } x \in \mathbb{Q} \\ \cos(x) & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   State whether the following sentences are true or false.
   (a) f is bounded.
   (b) $\max\{f(x) : x \in [0, 2\pi[\} = 1$.
   (c) $\min\{f(x) : x \in [0, 2\pi[\} = -1$.
   (d) $\lim_{x\to\frac{\pi}{2}} f(x) = 1$.
   (e) f is continuous.

(f) f is continuous at $x_0 = \frac{\pi}{4}$.

6. Compute the following limits if they exist.

   (a) $\lim_{x\to 1} \frac{x^3+x^2-2}{x-1}$

   (b) $\lim_{x\to 2} \frac{(-1)^{[x]}x^2+3}{x-2}$

   (c) $\lim_{x\to 0} \frac{\sin(x)^2}{\sin(x^2)}$

7. Let $A = \left\{ \left(\frac{\pi}{2} + n\pi\right)^{-1} : n \in \mathbb{N} \right\}$. Find, if it exists, a continuous extension of the function $f\colon]0,1] \setminus A \to \mathbb{R}$,

$$f(x) = \tan\left(\frac{1}{x}\right)\left(1 - \sin\left(\frac{1}{x}\right)^2\right)$$

   at the points $x_0 \in A \cup \{0\}$, or otherwise show that f cannot have a continuous extension at x_0.

8. Find the values $\alpha \in \mathbb{R}$ such that the limit $\lim_{x\to\alpha} \frac{x^4 - 2\alpha x^3 + 4x^2}{(x-\alpha)^2}$ exists in $\mathbb{R}$.

9. Study the continuity of the function $f\colon \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \dfrac{1}{1+2^{\frac{1}{x}}}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

   at $x = 0$.

10. Find the local and global maximum/minimum of the function $f(x) = |x^2 - x| + |x|$, by sketching the graph of the function.

11. Compute the following limits if they exist.

    (a) $\lim_{x\to 1} \frac{x^2-x}{x^2-2x+1}$

    (b) $\lim_{x\to +\infty} \left(\sqrt[3]{x+1} - \sqrt[3]{x}\right)$

    (c) $\lim_{x\to 0} \frac{(-1)^{[x]}}{\sin(x)^3} + \frac{1}{\sin(x)^2}$

12. Consider the function

$$f(x) = \frac{x(x-1)\tan(x-1)}{x^3 - 3x + 2},$$

    whose domain is $\mathbb{R} \setminus \{-2, 1\}$.

    (a) Study its continuity at $x_0 = 0$.

    (b) Find, if it exists, a continuous extension of the function f in $x_0 = 1$, or otherwise show that f cannot have a continuous extension at $x_0 = 1$.

13. (a) Prove or disprove that a function is continuous if and only if it is uniformly continuous.

    (b) Prove or disprove that $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ is a uniformly continuous function.

    (c) Show that the function $f\colon]0, b[\to \mathbb{R}$ defined by $f(x) = x^2$ is continuous and also uniformly continuous for $b < +\infty$. Show that f is not uniformly continuous when $b = +\infty$.

14. Let I be an interval, $f\colon I \to \mathbb{R}$ be a continuous function and $f(I)$ the image of I by f. Say if the following statement are true or false.

(a) $f(I)$ is an interval (where here we also admit the degenerate case $f(I) = [m, m] = \{m\}$).
(b) If I is a bounded and closed interval, then $f(I)$ is a bounded and closed interval (where here we also admit the degenerate case $f(I) = [m, m] = \{m\}$).
(c) If I is open, then $f(I)$ is an open interval.
(d) If $I = [a, b[$ with $a, b \in \mathbb{R}$, $a < b$, then f attains its maximum and minimum in I. That is, there exists $m, M \in R(f)$ such that $R(f) = [m, M]$.

15. Find, if it exists, continuous extension of the function $f :]0, 1] \to \mathbb{R}$ given by $f(x) = \frac{\tan(\sqrt{1+x}-1)}{x^{3/2}}$ at $x_0 = 0$, or otherwise show that f cannot have a continuous extension at x_0. (*Note: you have to care just about the limit from the right, that is:* $x \to 0^+$)

16. Use the intermediate value theorem to show that the following equations have at least one solution in $\mathbb{R}$:

    (a) $e^{x-1} = x + 1$
    (b) $x^2 - \dfrac{1}{x} = 1$

17. State if the following functions are continuous and differentiable at $x = 0$.

    (a) $|\sin(x)|$
    (b) $|x^3|$

18. Let f and g be two continuous functions in $[a, b]$, such that $f(a) > g(a)$ and $f(b) < g(b)$. Show that there is $c \in]a, b[$ such that $f(c) = g(c)$. (*Hint: use the function* $h = f - g$ *and the intermediate value theorem.*)

19. Find the inverse of the following functions if they exist. Give the domain of both functions.

    (a) $f(x) = \sqrt{(2x + 4)^3 - 7}$
    (b) $f(x) = \frac{2x+3}{3x+5}$
    (c) $f(x) = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}$

20. Study the continuity of the function $f : \mathbb{R} \to \mathbb{R}$ in $x = 0$, where

$$f(x) = \begin{cases} \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

21. State if the following statements are true or false:

    (a) If $\lim\limits_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = c \in \mathbb{R}$ then f is continuous at 0.
    (b) If $f \circ g$ is continuous, then f is continuous.
    (c) For every $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, there exists at most one value $a \in \mathbb{R}$ such that the function $\hat{f} : \mathbb{R} \to \mathbb{R}$

$$\hat{f}(x) := \begin{cases} f(x) & x \neq 0, \\ a & x = 0, \end{cases}$$

    is continuous.
    (d) If $|f(x)|$ is continuous everywhere then $f(x)$ is also continuous.
    (e) If $f \circ g$ is continuous, then g is continuous.
    (f) If functions f and g are continuous everywhere then f/g is also continuous everywhere.
    (g) If $f(x)$ is continuous everywhere, then $|f(x)|$ is continuous everywhere.
    (h) If the composition $f \circ g$ is not continuous at $x = a$, then g is not continuous at $x = a$ or f is not continuous at $g(a)$.