Analysis 1 - Exercise Set 4

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Compute
   (a) $(1+i\sqrt{3})^{1980}$
   (b) $(1+i\sqrt{3})^{1988}$

2. Find all the solution of the following equations in $\mathbb{C}$. [The unknown is $z = x + iy$, or, if you prefer you could use polar form.]
   (a) $z^2 = i$
   (b) $z^5 = 1$.
   (c) $z^2 = -3 + 4i$.

3. In the context of complex numbers, state if the following statements are true or false.
   (a) There exists a $n \in \mathbb{N}$ such that $(1+i\sqrt{3})^n$ is pure imaginary.
   (b) There exists a positive $n \in \mathbb{N}$ such that $(1-i\sqrt{3})^n$ is real.

4. Show that for all $\theta \in \mathbb{R}$ and for all $n \in \mathbb{N}$

$$(\cos(\theta) + i\sin(\theta))^n = (\cos(n\theta) + i\sin(n\theta)).$$

5. Prove that for all $z_1, z_2 \in \mathbb{C}$,
   (a) $z_1 = 0$ if and only if $|z_1| = 0$.
   (b) $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{i(\alpha_1 - \alpha_2)}$, where $z_2 \neq 0$ and

$$z_1 = |z_1|e^{i\alpha_1}, \quad z_2 = |z_2|e^{i\alpha_2}, \quad \alpha_1,\ \alpha_2 \in \mathbb{R}$$

   are the polar forms of the z_i using Euler's formula.
   (c) $|\frac{z_1}{|z_1|}| = 1$.

6. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $\mathrm{Im}(z(2-i)) = 1$ is
   (a) A point.
   (b) A line.
   (c) A circle.
   (d) Empty.

7. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $\bar{z} = i(z-1)$ is
   (a) A point.
   (b) A line.
   (c) A circle.

(d) Empty.

8. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $z^2 \cdot \bar{z} = z$ is
   (a) A point.
   (b) A circle.
   (c) A point and a circle.
   (d) A disk.

9. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $|z + 3i| = 3|z|$ is
   (a) A point.
   (b) A line.
   (c) A circle.
   (d) Empty.

10. Given the function $f \colon \mathbb{C} \to \mathbb{C}$ defined as $f(z) = \frac{1+iz}{iz+i}$
    (a) find the domain of the function f. That is, determine the set $\mathrm{Dom}(f) \subseteq \mathbb{C}$ such that $z \in \mathrm{Dom}(f)$ if and only if $f(z)$ is defined;
    (b) find all complex numbers z such that $f(z) = z$;
    (c) find the preimages of $3 + i$.

11. (a) Prove that for all $n \in \mathbb{N}$, $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$.
    (b) Find all the solutions to the equation $x^n = 1$, for $n \in \mathbb{N}$.

12. Find all the solutions of the following equations in $\mathbb{C}$.
    (a) $z^2 + 6z + 12 - 4i = 0$
    (b) $(z^3 - 1)^2 = -1$

13. (a) Let $\{a_n\}$ be a sequence and let $\{b_n\}$ be the sequence defined as $b_n := |a_n|$. Prove that $\{a_n\}$ is bounded if and only if so is $\{b_n\}$.
    (b) Prove that if $q < -1$, then the sequence $x_n = aq^n$, $a \in \mathbb{R}$, $a \neq 0$, is unbounded.
    (c) Provide an example of a sequence $\{a_n\}$ that is bounded above and such that $\{|a_n|\}$ is not bounded above.

14. Let $\{x_n\}$ be the recursive sequence defined as

$$\begin{cases} x_n = x_{n-1} + (-1)^n n^2 \\ x_0 = 0. \end{cases}$$

Prove that for all $m \in \mathbb{N}$

$$\begin{cases} x_{2m} = (2m+1)m \\ x_{2m+1} = -(2m+1)(m+1). \end{cases}$$

[*Hint*: use induction on m].
Is $\{x_n\}$ bounded from above? Is it bounded from below?