

Analysis 1 - Exercise Set 2

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Let $p \in \mathbb{N}$ be a prime number. Prove that \sqrt{p} is not rational.
 - (b) Show that $\sqrt{7 + \sqrt{17}}$ is irrational. (*Hint: Use part (a) to prove that $\sqrt{17}$ is irrational. Now assume that $\sqrt{7 + \sqrt{17}}$ is rational and show that it contradicts the fact that $\sqrt{17}$ is irrational.*)
 - (c) Show that $\sqrt{2} + \sqrt[3]{3}$ is irrational. (*Hint: Let $r = \sqrt{2} + \sqrt[3]{3}$ and assume it is rational. Compute $(r - \sqrt{2})^3$ and use the result that you obtained plus the assumption on the rationality of r to find a contradiction.*)
2. Let S be a subset of \mathbb{R} . Let a be a lower bound (respectively an upper bound) for S . Show that any real number b such that $b < a$ (respectively $b > a$) then b is also a lower bound (resp. an upper bound) for S .
3. Let A be a bounded interval in \mathbb{R} , i.e., A is a subset of \mathbb{R} of either one of the following forms: $[a, b]$, or $]a, b[$, or $[a, b[$, or $]a, b]$, with $a, b \in \mathbb{R}$ and $a < b$. State if the following statements are true or false. If you true, explain why. If false, find an example of an interval that contradicts that statement.
 - (a) $\sup(A) \in A$ and $\inf(A) \in A$.
 - (b) If $\sup(A) \in A$ and $\inf(A) \in A$ then A is closed.
 - (c) If A is closed then $\sup(A) \in A$ and $\inf(A) \in A$.
 - (d) If $\sup(A) \notin A$ and $\inf(A) \notin A$ then A is open.
 - (e) If A is open then $\sup(A) \notin A$ and $\inf(A) \notin A$.
4. Let A be a bounded interval in \mathbb{R} , i.e., A is a subset of \mathbb{R} of either one of the following forms: $[a, b]$, or $]a, b[$, or $[a, b[$, or $]a, b]$, with $a, b \in \mathbb{R}$ and $a < b$. Show that $\inf A = a$, $\sup A = b$. When is the infimum (resp. maximum) of A a minimum (resp. a maximum)?
5. Let S be a subset of \mathbb{R} . Show that if $\sup(S)$, $\inf(S)$, $\max(S)$, $\min(S)$ exist, then they are unique.
6. Let $S \subseteq \mathbb{R}$ be the subset of the real numbers defined as $S := \{x \in \mathbb{R} \mid x \in \mathbb{Q} \text{ and } x^3 \geq 5\}$.
 - (a) Show that S is not empty (i.e., exhibit an element of S).
 - (b) Show that $\sqrt[3]{5}$ is a lower bound for S .
 - (c) Show that $\inf(S) = \sqrt[3]{5}$. (*Hint: you should use the denseness of \mathbb{Q} in this step.*)

Hint: you can use the fact that every real number has a unique real cubic root, and that $a^3 \leq b^3$ if and only if $a \leq b$.
7. For each of the following sets, check if they are bounded or unbounded. When the set is bounded from above or below, give a few examples of lower and upper bounds, then compute the supremum and infimum and check if maximum and minimum exist.
 - (a) $A = \{x \in \mathbb{R} \mid x^2 \leq 2\}$.

(b) $B = \{x \in \mathbb{R} \mid x \in \mathbb{Q} \text{ and } x^2 \leq 2\}$.

(c) $C = \{(-1)^n + \frac{1}{n+1} \mid n \in \mathbb{N}\}$.

8. Let a be a real number. Assume that $a \geq 0$. Prove that $a = 0$ if and only if for any $\varepsilon > 0$, $a \leq \varepsilon$.

9. Let L and L' be two real numbers. Prove that the following are equivalent:

(a) $L = L'$;

(b) for every $\varepsilon > 0$, $|L - L'| \leq \varepsilon$.

10. Let $S \subseteq \mathbb{R}$ be a non-empty subset. Assume that S is bounded from above and that $\sup(S) \notin S$. Show that the following fact holds: for every $\varepsilon > 0$, $S \cap]\sup(S) - \varepsilon, \sup(S)[$ is infinite (i.e., there are infinitely many elements of S in $] \sup(S) - \varepsilon, \sup(S)[$).

Hint: In this problem, you can freely use that a finite non-empty set has both maximum and minimum.

11. Let S be a non-empty and bounded subset of \mathbb{R} . We define

$$S' := \{x \in \mathbb{R} \mid -x \in S\}.$$

Show that

(a) If M is an upper bound of S , then $-M$ is a lower bound of S' .

(b) If m is a lower bound of S , then $-m$ is an upper bound of S' .

(c) $\sup(S) = -\inf(S')$.

(d) $\inf(S) = -\sup(S')$.

12. Let S be the subset of \mathbb{R} defined as

$$S := \bigcap_{n=1}^{\infty} [0, 1 + \frac{1}{n}]$$

Compute $m := \sup S$. Is m the maximum of S ? (*Hint: $x \in S \iff \forall n \in \mathbb{N}, x \in [0, 1 + \frac{1}{n}]$*)

13. (Multiple choice) The subset S of \mathbb{R}^2 defined as¹

$$S := \{(x, y) \in \mathbb{R}^2 \mid x = -y, -y = x - 1\}$$

is:

(a) A point.

(b) A line.

(c) A circle.

(d) Empty.

14. (Multiple choice) The subset S of \mathbb{R}^2 defined as

$$S := \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + (y+3)^2} = 3\sqrt{x^2 + y^2}\}$$

is:

(a) A point.

(b) A line.

(c) A circle.

(d) Empty.

¹In this exercise (x, y) does not denote an open interval between x and y , but it instead denotes the point of coordinates x and y in \mathbb{R}^2