

Analysis 1 - Exercise Set 1

1. Real Numbers.

- (a) Explain the difference between a rational and an irrational number.
- (b) Classify the following numbers as rational, irrational, natural, integer. (A number may belong to more than one set).
- (i) -2
 - (ii) $4\frac{1}{3}$
 - (iii) $\sqrt{10}$
 - (iv) 0
 - (v) π
- (c) Show that $\sqrt{7}$ is an irrational number. (*Hint: assume that you can write $\sqrt{7} = \frac{a}{b}$ where a and b are integers where their greatest common divisor is $\gcd(a, b) = 1$. Then try to find a contradiction.*)

2. Trigonometry.

Show that the following identities hold:

- (a) $\sin^6 x + \cos^6 x = 1 - 3 \sin^2 x \cos^2 x$
 - (b) $\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$
 - (c) $\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$
 - (d) $\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$
 - (e) $\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$
- (*Hints: For (a) use the identities: $(a^3+b^3) = (a+b)(a^2-ab+b^2)$, $(a+b)^2 = a^2+2ab+b^2$, $\sin^2 x + \cos^2 x = 1$. For (b)-(e) use the identities*

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

and compute $\sin(\alpha + \beta) + \sin(\alpha - \beta)$, $\sin(\alpha + \beta) - \sin(\alpha - \beta)$ etc. then try to find x and y in terms of α and β)

3. Trigonometry.

Simplify the following trigonometric expressions to obtain algebraic expressions (i.e., only involving sums, ratios, roots, etc.).

Example: if $-1 \leq x \leq 1$, we have $\cos(\arcsin x) = \sqrt{1 - x^2}$.

- (a) $\sin(\arcsin x)$, where $-1 \leq x \leq 1$
- (b) $\sin(\arccos x)$, where $-1 \leq x \leq 1$
- (c) $\tan(\arccos x)$, where $-1 \leq x \leq 1$

4. Arithmetic manipulations.

Prove the following identities.

(a)

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \quad n \geq 1.$$

(Hint: try to add the elements of the two finite sequences $(1, 2, \dots, n)$ and $(n, n-1, n-2, \dots, 1)$ term by term)

(b) Give an alternative proof, to the one given in the first lecture, for the equality

$$a + a^2 + a^3 + \cdots + a^n = a \cdot \frac{1 - a^n}{1 - a}, \quad a \neq 1, \quad n > 1.$$

(Hint: use the identity

$$(b^n - a^n) = (b - a)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}),$$

and replace b with 1)

(c)* For any $n \in \mathbb{N}$,

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

(Hint: start by computing $k^3 - (k-1)^3$ for a natural number k)

5. Equations.

Solve the following equations:

(a) $\frac{2x}{x+1} = \frac{2x-1}{x}$;

(b) $x^4 - 3x^2 + 2 = 0$;

(c) $3|x-4| = 10$.

6. Inequalities.

Determine the solutions to the following inequalities.

(a) $x^2 < 2x + 8$;

(b) $x(x-1)(x-2) > 0$;

(c) $\frac{2x-3}{x+1} \leq 1$;

(d) $|x^2 - 1| \leq 1$.

7. Functions.

Let

$$f(x) = \frac{1}{1 - \frac{2}{1 + \frac{1}{1-x}}}$$

(a) Find x , such that $f(x) = 3$.

(b) Find the domain of f .

8. Functions.

Do there exist functions f and g defined on \mathbb{R} such that

$$f(x) + g(y) = xy$$

for all real numbers x and y ? (Hint: Try to evaluate for $(x, y) = (0, 0)$, $(x, y) = (1, 0)$, $(x, y) = (0, 1)$, $(x, y) = (1, 1)$)

9. Functions.

Recall that a function $F: X \rightarrow Y$ is called injective if for every pair of elements a and b in X , $F(a) = F(b)$ implies that $a = b$; in other words, it is injective if distinct elements have distinct images.

Consider now three functions $f, g, h: \mathbb{R} \rightarrow \mathbb{R}$. For each of the following statements, say whether that is true or false. If you think it is true, then provide a proof of that, or, else, if false, provide a counterexample.

- (a) $f \circ (g + h) = f \circ g + f \circ h$;
- (b) $(f + g) \circ h = (f \circ h) + (g \circ h)$;
- (c) $f \circ g = g \circ f$ if and only if $f = g$;
- (d) if $f \circ f$ is injective then f is injective;
- (e) if $f \circ g$ is injective then g is injective;
- (f) if $f \circ g$ is injective then f is injective.

10. Functions.

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ and $g: \mathbb{N} \rightarrow \mathbb{N}$ be defined by $f(n) = n^2$ and $g(n) = n + 1$, respectively.

- (a) Compute $f \circ g$;
- (b) compute $g \circ f$;
- (c) compute g^m for every $m \in \mathbb{N}$.

11. * Functions.

Consider the following set of $n + 1$ points in \mathbb{R}^2 :

$$S := \{(x_i, y_i) | i = 0, 1, \dots, n\},$$

where $x_i \neq x_j$ for $i \neq j$.

Find a polynomial p of degree at most n such that $p(x_i) = y_i$ for any $i = 0, 1, \dots, n$.

You may use the following fact: If p is a sum of polynomials of degree n , then p is a polynomial of degree at most n .

(Hint: Try to first find degree n polynomials φ_i for $i = 0, 1, \dots, n$ s.t. $\varphi_i(x_j) = \delta_{ij}$, where δ_{ij} is defined as follows:

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases}$$

Using the polynomials φ_i , can you construct p ?)

12. Sets.

For sets E, F and D prove the following:

- (a) Commutativity: $E \cap F = F \cap E$ and $E \cup F = F \cup E$;
- (b) Associativity: $D \cap (E \cap F) = (D \cap E) \cap F$ and $D \cup (E \cup F) = (D \cup E) \cup F$;
- (c) Distributivity: $D \cap (E \cup F) = (D \cap E) \cup (D \cap F)$ and $D \cup (E \cap F) = (D \cup E) \cap (D \cup F)$;
- (d) De Morgan laws: $(E \cap F)^c = E^c \cup F^c$ and $(E \cup F)^c = E^c \cap F^c$.

13. Representations of numbers.

Prove that a real number is a rational number if and only if the decimal representation becomes periodic.