













Ens: Thomas Mountford  
 Analyse I - Section  
 13 January 2025  
 3h30

# Student

SCIPER: **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 16 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  - +3 points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - 1 points if your answer is incorrect.
- For the **true/false** questions, we give :
  - +1 points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - 1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes   Read these guidelines   Beachten Sie bitte die unten stehenden Richtlinien		
choisir une réponse   select an answer Antwort auswählen	ne PAS choisir une réponse   NOT select an answer NICHT Antwort auswählen	Corriger une réponse   Correct an answer Antwort korrigieren
  		 
ce qu'il ne faut <b>PAS</b> faire   what should <b>NOT</b> be done   was man <b>NICHT</b> tun sollte		
     		

**First part: multiple choice questions**

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

**Question 1 :** The series

$$\sum_{n=1}^{\infty} \frac{e^{\lambda n}}{n^{1+\lambda}}$$

converges if and only if  $\lambda \in I$ , where  $I$  is the set

- $] -\infty, 0[$         $[-1, +\infty[$         $] -\infty, -1[$         $] -\infty, 0]$

**Question 2 :** Consider the equation

$$\frac{|z|}{z} = \frac{z^2}{4(\cos(\frac{\pi}{3}) + i \sin(\frac{\pi}{3}))}$$

Among the complex numbers given below, which one is a solution of the equation ?

- $z = 2(\cos(\frac{7\pi}{12}) + i \sin(\frac{7\pi}{12}))$         $z = 2(\cos(\frac{7\pi}{9}) + i \sin(\frac{7\pi}{9}))$   
  $z = \sqrt[3]{4}(\cos(\frac{13\pi}{12}) + i \sin(\frac{13\pi}{12}))$         $z = \sqrt[3]{4}(\cos(\frac{\pi}{9}) + i \sin(\frac{\pi}{9}))$

**Question 3 :** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the bijective function given by

$$f(x) = x^3 + 3x + 1$$

and let  $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$  be its inverse. Then  $(f^{-1})'(1)$  is equal to

- $\frac{1}{3}$         $\frac{1}{5}$         $\frac{1}{6}$         $\frac{1}{78}$

**Question 4 :** Let  $f: [-1, 2] \rightarrow \mathbb{R}$  be defined by  $f(x) = e^{x+1}(x^2 - 2x + 1)$ . Then, its range,  $f([-1, 2])$  is

- $[0, e^3]$         $[4, e^3]$         $[0, 4]$         $[0, +\infty[$

**Question 5 :** The integral  $\int_{-1}^1 \frac{1}{x^2 - 4} dx$  equals

- $-\frac{1}{4} \log(3)$         $2 \log(3)$         $\frac{1}{2} \log(3)$         $-\frac{1}{2} \log(3)$

**Question 6 :** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = \begin{cases} \frac{\sin(x^2)}{x} & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

Then, at  $x = 0$ ,  $f$  is

- right differentiable but not left differentiable  
 left differentiable but not right differentiable  
 is both left differentiable and right differentiable but is not differentiable  
 differentiable

## CORRECTION

**Question 7 :** Let  $f: ]0, 1[ \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = \frac{\log(|\log(x)|)}{\log(x)}.$$

Then

- $\lim_{x \rightarrow 0^+} f(x) = 0$ 
  $\lim_{x \rightarrow 0^+} f(x) = -\infty$   
  $\lim_{x \rightarrow 0^+} f(x) = +\infty$ 
  $\lim_{x \rightarrow 0^+} f(x) = -1$

**Question 8 :** The integral  $\int_1^{+\infty} \frac{\log(t)}{t^2} dt$

- converges and is equal to 2
 converges and is equal to  $\frac{1}{2}$   
 diverges
 converges and is equal to 1

**Question 9 :** The limit

$$\lim_{n \rightarrow \infty} \left( \frac{n^2 + 1}{n(n + 3)} \right)^n$$

exists and equals

- $e^{-3}$ 
  $e^{-1}$ 
 0
 1

**Question 10 :** Let  $A \subset \mathbb{R}$  be the set defined by

$$A = \left\{ x \in \mathbb{R}^* : \frac{1}{x} \geq 2 \right\}.$$

Then

- $\inf A = 0$ 
  $\inf A = \frac{1}{2}$   
  $\inf A = 2$ 
  $A$  is not bounded from below

**Question 11 :** The radius of convergence of the power series  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$  is

- $\frac{1}{4}$ 
  $+\infty$ 
 4
 0

**Question 12 :** Let  $(u_n)_{n \geq 0}$  be the sequence defined by  $u_0 = \sqrt{3}$  and for  $n \geq 1$ ,  $u_n = \sqrt{3u_{n-1}}$ . Then

- $\lim_{n \rightarrow \infty} u_n = \sqrt{3}$ 
  $\lim_{n \rightarrow \infty} u_n = 3$ 
  $(u_n)_{n \geq 0}$  diverge
  $\lim_{n \rightarrow \infty} u_n = 0$

**Question 13 :** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = \begin{cases} \frac{x}{3} & \text{if } x \leq 0 \\ \frac{\log(x^2 + 1)}{3x} & \text{if } x > 0 \end{cases}$$

Then

- $f$  is continuously differentiable on  $\mathbb{R}$   
  $f$  is differentiable but not continuously differentiable on  $\mathbb{R}$   
  $f$  is not continuous on  $\mathbb{R}$   
  $f$  is continuous but not differentiable on  $\mathbb{R}$

## CORRECTION

**Question 14 :** Let  $f(x) = \log\left(\frac{3}{2} + x\right)$ . Then, the Taylor expansion of order 2 of  $f$  around  $x_0 = 0$  is given by

$\log\left(\frac{3}{2}\right) + \frac{2}{3}x + \frac{2}{9}x^2 + x^2\varepsilon_2(x)$

$\log\left(\frac{1}{2}\right) + x - \frac{x^2}{2} + x^2\varepsilon_2(x)$

$\log\left(\frac{3}{2}\right) + \log\left(\frac{3}{2}\right)x - \frac{\log\left(\frac{3}{2}\right)}{2}x^2 + x^2\varepsilon_2(x)$

$\log\left(\frac{3}{2}\right) + \frac{2}{3}x - \frac{2}{9}x^2 + x^2\varepsilon_2(x)$

**Question 15 :** The series  $\sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k^3 - k}}$

 does not converge but converges absolutely

 converges but does not converge absolutely

 converges and converges absolutely

 does not converge and does not converge absolutely

**Question 16 :**

Let  $f: [0, +\infty[ \rightarrow \mathbb{R}$  be a continuous function and let  $(a_k)_{k \geq 0}$ ,  $(b_k)_{k \geq 0}$  be two sequences defined by

$$a_k = f(2k) \quad \text{and} \quad b_k = f(2k + 1).$$

If

$$\lim_{k \rightarrow \infty} a_k = -1, \quad \text{and} \quad \lim_{k \rightarrow \infty} b_k = 1,$$

then, the equation  $f(x) = 0$

 has an infinite amount of solutions

 has exactly one solution

 has exactly two solutions

 has no solution

**Question 17 :** Let  $(a_n)_{n \geq 0}$  be a sequence such that  $\lim_{n \rightarrow \infty} a_n = 2$ , and let  $(b_n)_{n \geq 0}$  be the sequence given by

$$b_n = 1 + a_n \cos\left(n\frac{\pi}{2}\right), \quad n \geq 0.$$

Then

  $\liminf_{n \rightarrow \infty} b_n = -1$  and  $\limsup_{n \rightarrow \infty} b_n = 3$ 
  $\liminf_{n \rightarrow \infty} b_n = 3$  and  $\limsup_{n \rightarrow \infty} b_n = 3$ 
  $\liminf_{n \rightarrow \infty} b_n = 1$  and  $\limsup_{n \rightarrow \infty} b_n = 3$ 
  $\liminf_{n \rightarrow \infty} b_n = -2$  and  $\limsup_{n \rightarrow \infty} b_n = 2$ 

**Question 18 :** The integral  $\int_0^1 \frac{x}{\sqrt{x+1}} dx$  equals

$\frac{8 - 2\sqrt{2}}{3}$

$\frac{4 - 2\sqrt{2}}{3}$

$4 - 2\sqrt{2}$

$\frac{\sqrt{2} + 1}{3}$

**Second part: true/false questions**

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

**Question 19 :** Let  $(a_n)_{n \geq 0}$  be a bounded sequence such that for all  $n \in \mathbb{N}$ ,  $a_n > 3$ . Then  $\liminf_{n \rightarrow \infty} a_n > 3$ .

TRUE       FALSE

**Question 20 :** There exists a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  which is both bounded and injective.

TRUE       FALSE

**Question 21 :** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable function. Then

$$\frac{1}{2}f(t)^2 = \frac{1}{2}f(0)^2 + \int_0^t f(x)f'(x) dx \quad \forall t \in \mathbb{R}.$$

TRUE       FALSE

**Question 22 :** If the power series  $\sum_{k=1}^{\infty} a_k (x-1)^k$  converges at  $x=0$ , then it converges for  $x=2$ .

TRUE       FALSE

**Question 23 :** Let  $f$  and  $g$  be two functions with Taylor expansion of order 1 around  $x_0 = 0$  given by

$$f(x) = 1 + 2x + x\varepsilon(x),$$

$$g(x) = 1 + \frac{1}{2}x + x\varepsilon(x).$$

Then, the Taylor expansion of order 1 of  $f(g(x))$  around  $x_0 = 0$  is given by

$$f(g(x)) = 3 + x + x\varepsilon(x).$$

TRUE       FALSE

**Question 24 :** The roots of the polynomial  $z^4 + z^3 - 2z^2 + 2z + 4$  are  $\{-2, -1, \frac{1}{4}, 1+i\}$ .

TRUE       FALSE

**Question 25 :** Let  $f: ]0, 1[ \rightarrow \mathbb{R}$  be a continuous function. Then there exists a continuous function  $g: [0, 1] \rightarrow \mathbb{R}$  such that  $g(x) = f(x)$  for all  $x \in ]0, 1[$ .

TRUE       FALSE

## CORRECTION

**Question 26 :** Let  $A, B \subset \mathbb{R}$  be two bounded nonempty sets and  $c \in \mathbb{R}$ . Then,

$$\sup\{x + c : x \in A\} - \sup\{x + c : x \in B\} = \sup A - \sup B.$$

TRUE       FALSE

**Question 27 :** Let  $f: \mathbb{R} \rightarrow ]0, \infty[$  be a differentiable function. Then the function  $g: \mathbb{R} \rightarrow ]0, \infty[$  defined by  $g(x) = f(x)^{f(x)}$  is also differentiable on  $\mathbb{R}$ .

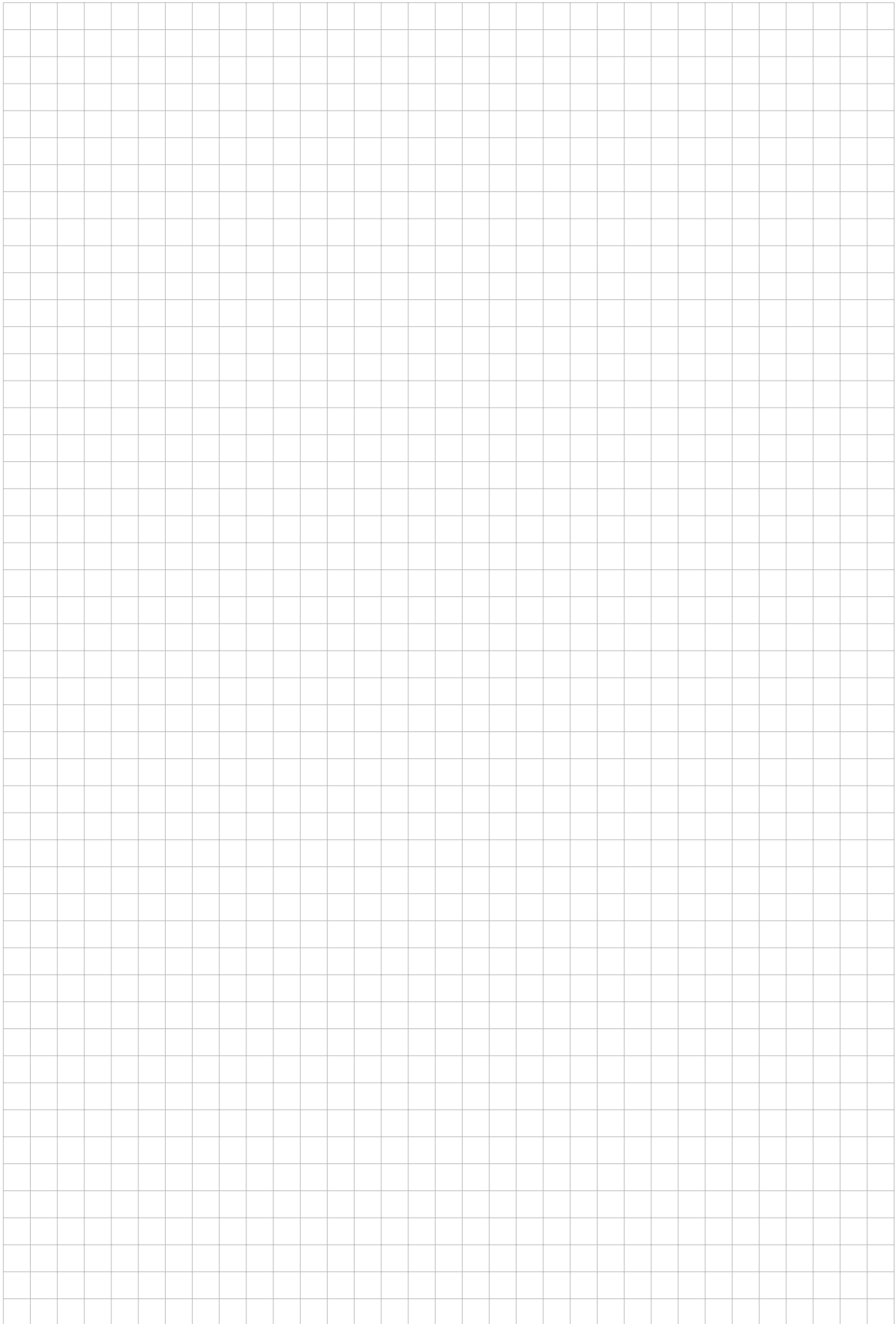
TRUE       FALSE

**Question 28 :** Let  $(a_n)_{n \geq 0}$  and  $(b_n)_{n \geq 0}$  be two sequences such that for all  $n \in \mathbb{N}$ ,  $0 < a_n < b_n$ . If the series  $\sum_{n=0}^{\infty} a_n$  diverges, then the series  $\sum_{n=0}^{\infty} \frac{1}{b_n}$  converges.

TRUE       FALSE



CORRECTION



CORRECTION

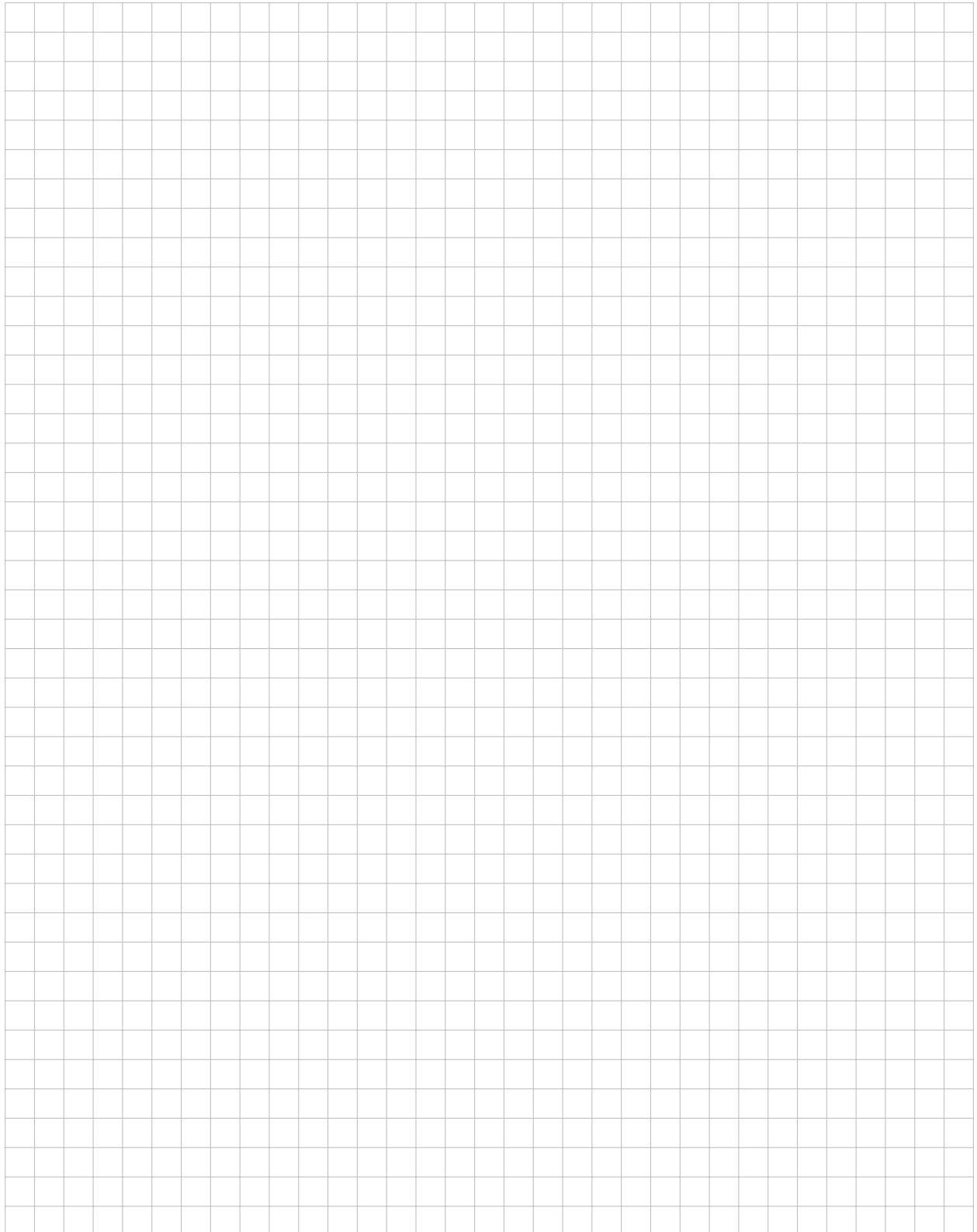
**Question 30:** *This question is worth 4 points.*

0  1  2  3  4

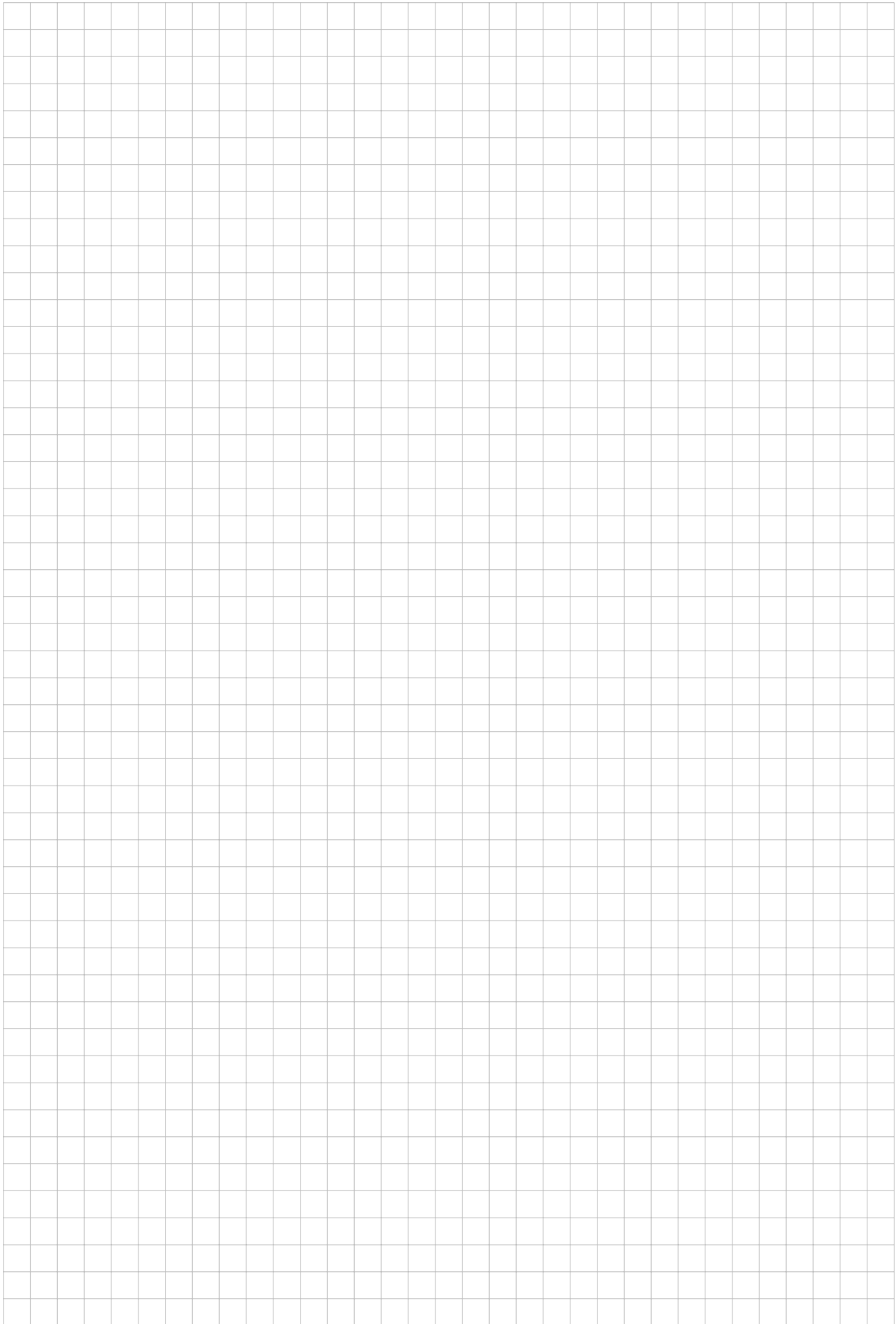
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Show by induction or otherwise that

$$\forall n \in \mathbb{N}^*, \sum_{k=1}^n k(k+1) = \frac{1}{3}n(n+1)(n+2).$$



CORRECTION



CORRECTION

**Question 31:** *This question is worth 5 points.*

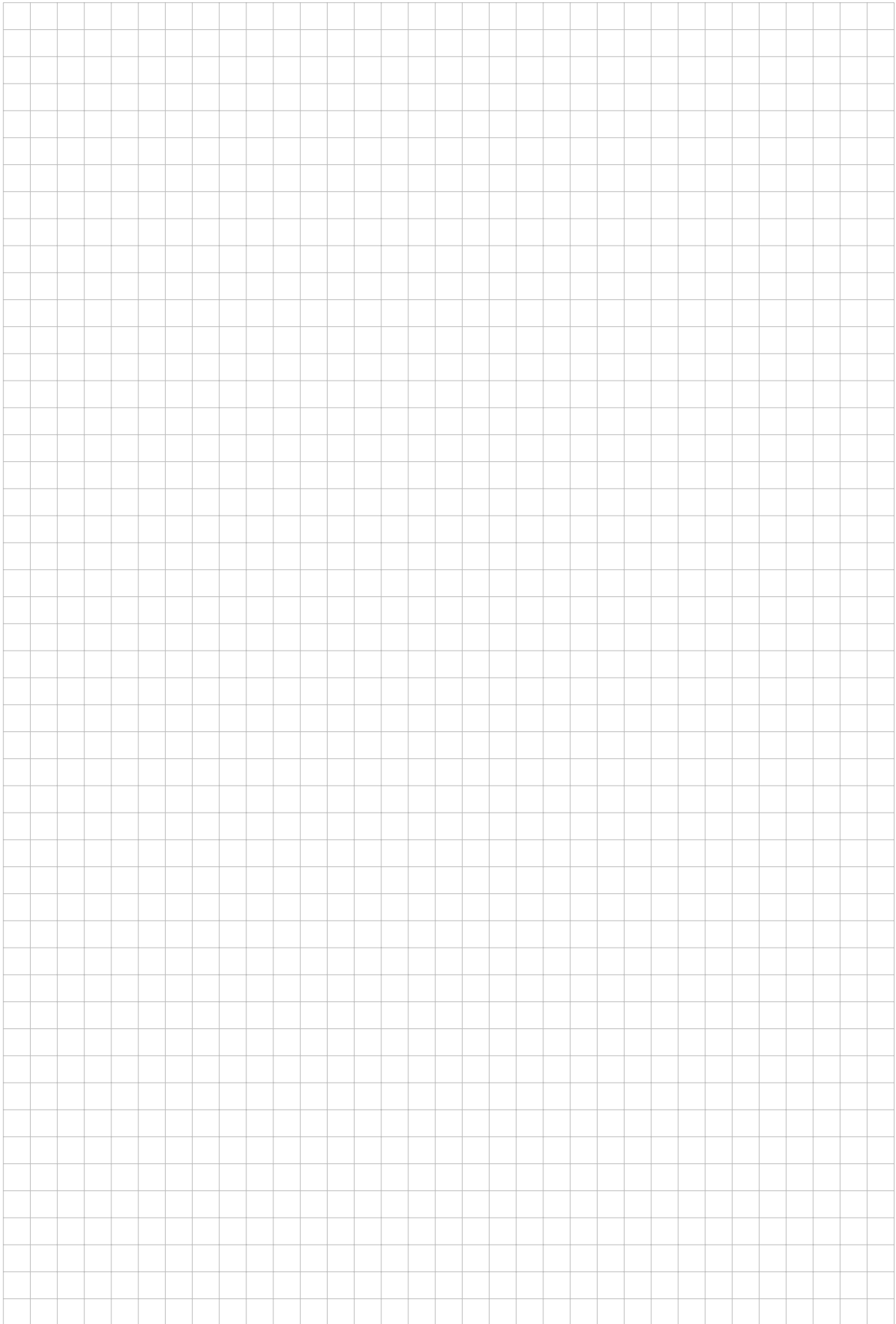
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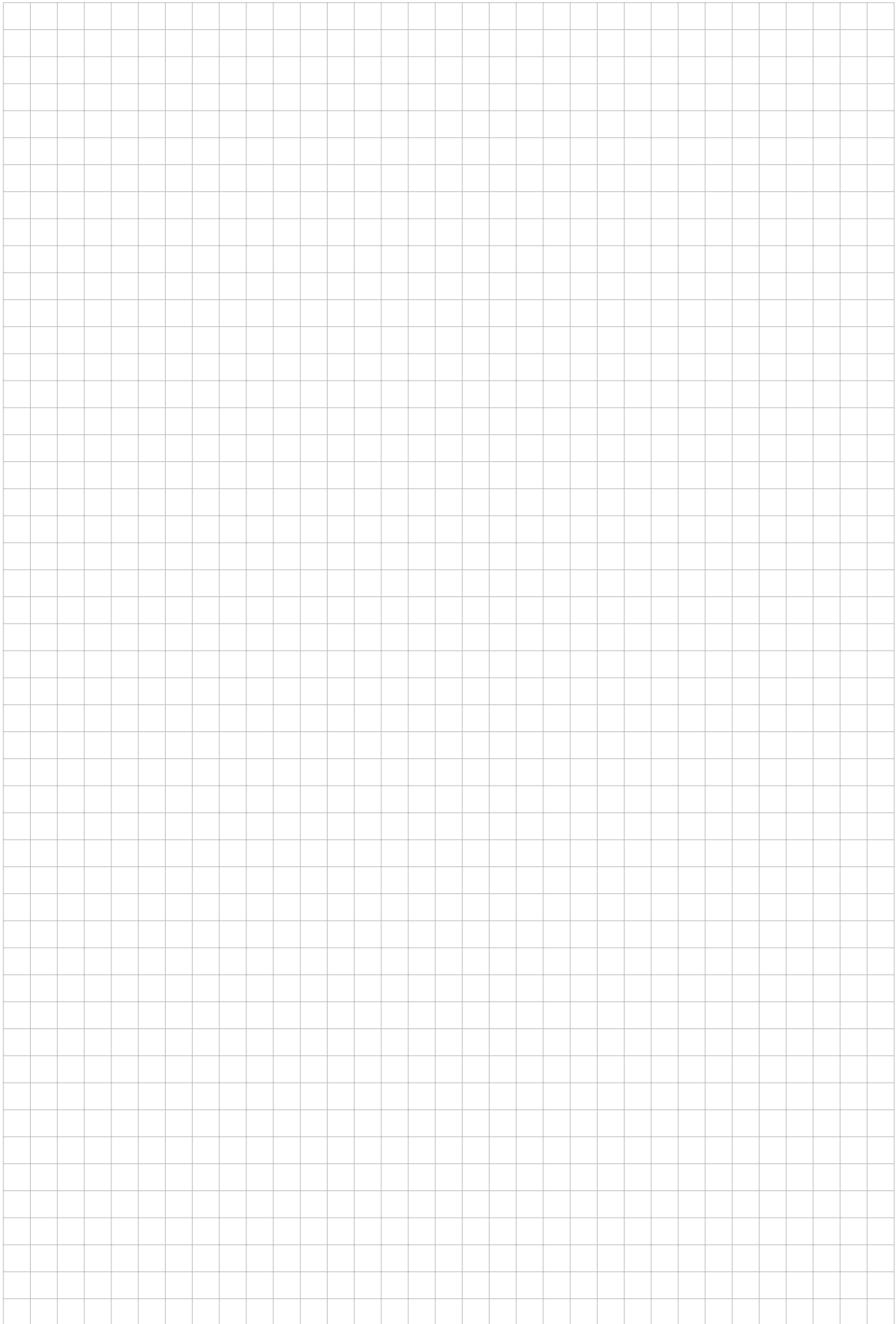
- (a) Let  $(a_n)_{n \geq 1}$  be a sequence and  $l \in \mathbb{R}$ . Give the definition of "The sequence  $(a_n)_{n \geq 1}$  converges to  $l$ ".
- (b) Let  $(x_n)_{n \geq 1}$  be a sequence such that both subsequences  $(x_{2k})_{k \geq 1}$  and  $(x_{2k+1})_{k \geq 1}$  converge to the same limit  $l$ . Show that  $(x_n)_{n \geq 1}$  converges to  $l$ .



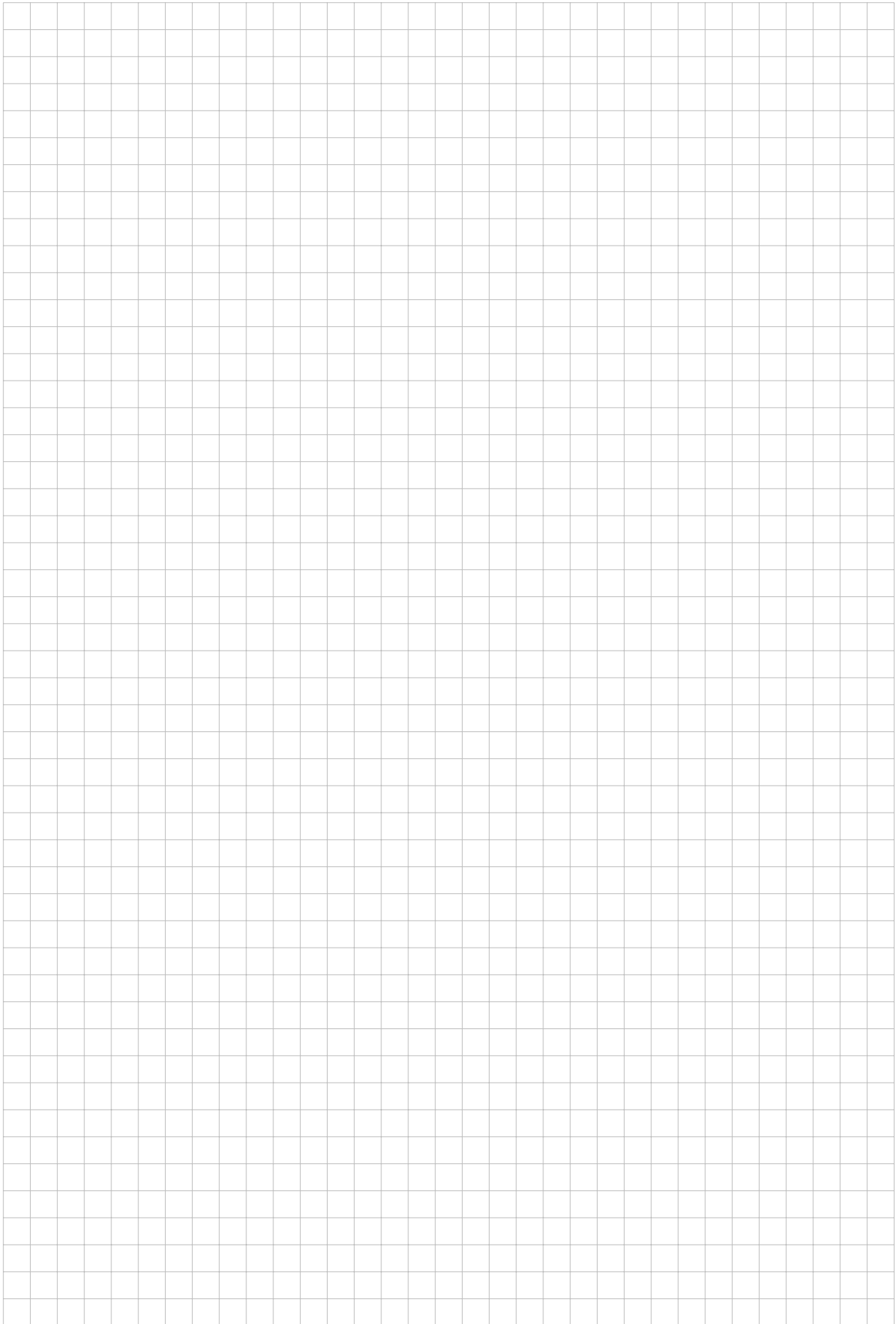
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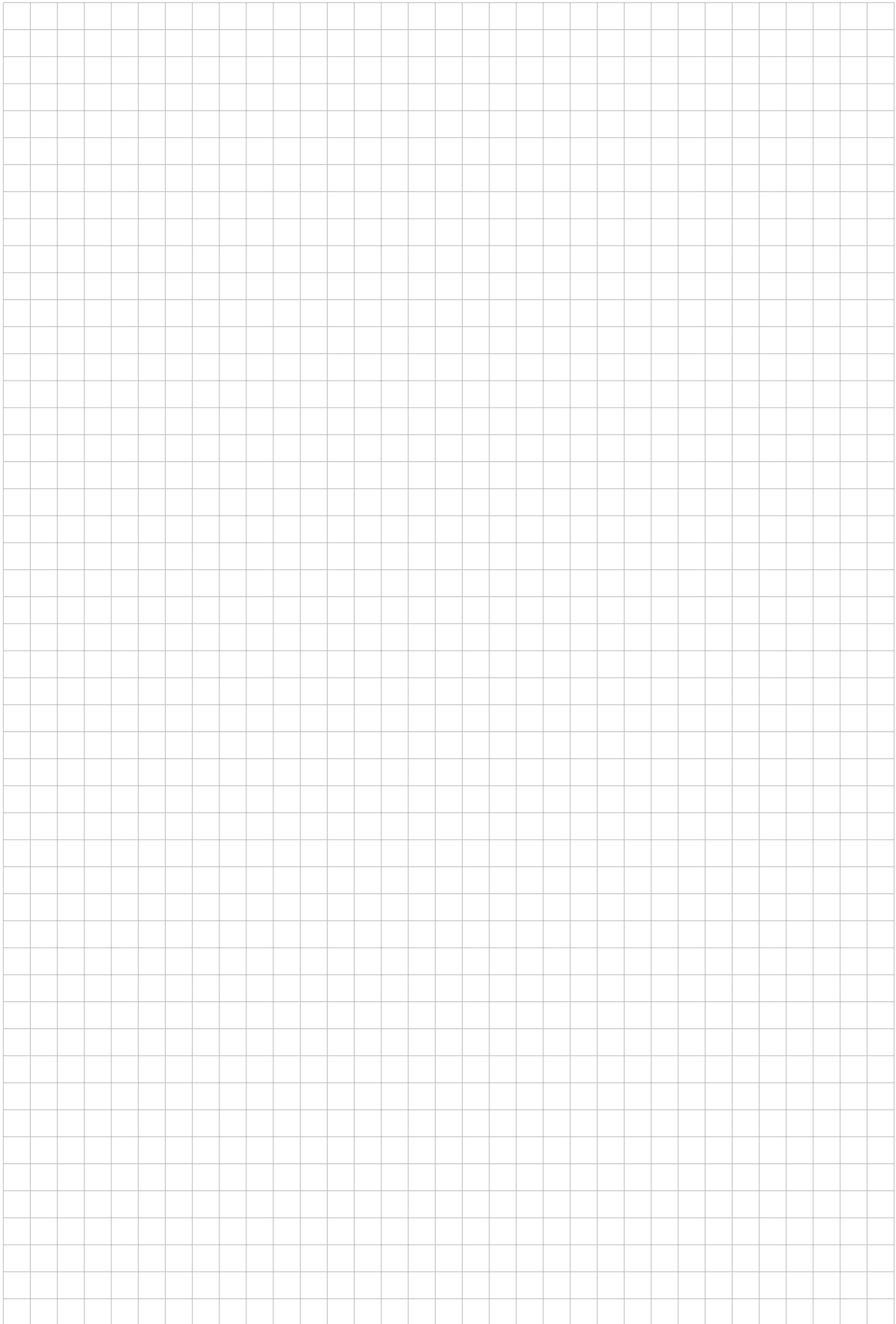
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