Ens: Thomas Mountford
Analyse I - Section
13 January 2025
3h30

XXX

Student

SCIPER : **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 16 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  $+3$ points if your answer is correct,
  0 points if you give no answer or more than one,
  -1 points if your answer is incorrect.
- For the **true/false** questions, we give :
  $+1$ points if your answer is correct,
  0 points if you give no answer or more than one,
  -1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : The series

$$\sum_{n=1}^{\infty} \frac{e^{\lambda n}}{n^{1+\lambda}}$$

converges if and only if $\lambda \in I$, where I is the set

- ☐ $]-\infty, 0[$
- ☐ $[-1, +\infty[$
- ☐ $]-\infty, -1[$
- ☐ $]-\infty, 0]$

Question 2 : Consider the equation

$$\frac{|z|}{z} = \frac{z^2}{4(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right))}.$$

Among the complex numbers given below, which one is a solution of the equation ?

- ☐ $z = 2(\cos\left(\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(\frac{7\pi}{12}\right))$
- ☐ $z = 2(\cos\left(\frac{7\pi}{9}\right) + \mathrm{i}\sin\left(\frac{7\pi}{9}\right))$
- ☐ $z = \sqrt[3]{4}(\cos\left(\frac{13\pi}{12}\right) + \mathrm{i}\sin\left(\frac{13\pi}{12}\right))$
- ☐ $z = \sqrt[3]{4}(\cos\left(\frac{\pi}{9}\right) + \mathrm{i}\sin\left(\frac{\pi}{9}\right))$

Question 3 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the bijective function given by

$$f(x) = x^3 + 3x + 1$$

and let $f^{-1}\colon \mathbb{R} \to \mathbb{R}$ be its inverse. Then $(f^{-1})'(1)$ is equal to

- ☐ $\frac{1}{3}$
- ☐ $\frac{1}{5}$
- ☐ $\frac{1}{6}$
- ☐ $\frac{1}{78}$

Question 4 : Let $f\colon [-1, 2] \to \mathbb{R}$ be defined by $f(x) = e^{x+1}(x^2 - 2x + 1)$. Then, its range, $f([-1, 2])$ is

- ☐ $[0, e^3]$
- ☐ $[4, e^3]$
- ☐ $[0, 4]$
- ☐ $[0, +\infty[$

Question 5 : The integral $\displaystyle\int_{-1}^{1} \frac{1}{x^2 - 4}\,\mathrm{d}x$ equals

- ☐ $-\frac{1}{4}\log(3)$
- ☐ $2\log(3)$
- ☐ $\frac{1}{2}\log(3)$
- ☐ $-\frac{1}{2}\log(3)$

Question 6 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{\sin(x^2)}{x} & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

Then, at $x = 0$, f is

- ☐ right differentiable but not left differentiable
- ☐ left differentiable but not right differentiable
- ☐ is both left differentiable and right differentiable but is not differentiable
- ☐ differentiable

Question 7 : Let $f\colon\ ]0,1[\ \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{\log(|\log(x)|)}{\log(x)}.$$

Then

- ☐ $\lim_{x\to 0^+} f(x) = 0$
- ☐ $\lim_{x\to 0^+} f(x) = -\infty$
- ☐ $\lim_{x\to 0^+} f(x) = +\infty$
- ☐ $\lim_{x\to 0^+} f(x) = -1$

Question 8 : The integral $\displaystyle\int_1^{+\infty} \frac{\log(t)}{t^2}\,\mathrm{d}t$

- ☐ converges and is equal to 2
- ☐ converges and is equal to $\frac{1}{2}$
- ☐ diverges
- ☐ converges and is equal to 1

Question 9 : The limit

$$\lim_{n\to\infty} \left(\frac{n^2+1}{n(n+3)}\right)^n$$

exists and equals

- ☐ e^{-3}
- ☐ e^{-1}
- ☐ 0
- ☐ 1

Question 10 : Let $A \subset \mathbb{R}$ be the set defined by

$$A = \left\{x \in \mathbb{R}^* \ :\ \frac{1}{x} \geq 2\right\}.$$

Then

- ☐ $\inf A = 0$
- ☐ $\inf A = \frac{1}{2}$
- ☐ $\inf A = 2$
- ☐ A is not bounded from below

Question 11 : The radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}x^n$ is

- ☐ $\frac{1}{4}$
- ☐ $+\infty$
- ☐ 4
- ☐ 0

Question 12 : Let $(u_n)_{n\geq 0}$ be the sequence defined by $u_0 = \sqrt{3}$ and for $n \geq 1$, $u_n = \sqrt{3u_{n-1}}$. Then

- ☐ $\lim_{n\to\infty} u_n = \sqrt{3}$
- ☐ $\lim_{n\to\infty} u_n = 3$
- ☐ $(u_n)_{n\geq 0}$ diverge
- ☐ $\lim_{n\to\infty} u_n = 0$

Question 13 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{x}{3} & \text{if } x \leq 0 \\[2ex] \dfrac{\log(x^2+1)}{3x} & \text{if } x > 0 \end{cases}$$

Then

- ☐ f is continuously differentiable on $\mathbb{R}$
- ☐ f is differentiable but not continuously differentiable on $\mathbb{R}$
- ☐ f is not continuous on $\mathbb{R}$
- ☐ f is continuous but not differentiable on $\mathbb{R}$

Question 14 : Let $f(x) = \log\left(\frac{3}{2} + x\right)$. Then, the Taylor expansion of order 2 of f around $x_0 = 0$ is given by

- ☐ $\log\left(\frac{3}{2}\right) + \frac{2}{3}x + \frac{2}{9}x^2 + x^2\varepsilon_2(x)$
- ☐ $\log\left(\frac{1}{2}\right) + x - \frac{x^2}{2} + x^2\varepsilon_2(x)$
- ☐ $\log\left(\frac{3}{2}\right) + \log\left(\frac{3}{2}\right)x - \frac{\log\left(\frac{3}{2}\right)}{2}x^2 + x^2\varepsilon_2(x)$
- ☐ $\log\left(\frac{3}{2}\right) + \frac{2}{3}x - \frac{2}{9}x^2 + x^2\varepsilon_2(x)$

Question 15 : The series $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k^3 - k}}$

- ☐ does not converge but converges absolutely
- ☐ converges but does not converge absolutely
- ☐ does not converge and does not converge absolutely
- ☐ converges and converges absolutely

Question 16 :
Let $f\colon [0, +\infty[\to \mathbb{R}$ be a continuous function and let $(a_k)_{k\geq 0}$, $(b_k)_{k\geq 0}$ be two sequences defined by

$$a_k = f(2k) \quad \text{and} \quad b_k = f(2k+1).$$

If

$$\lim_{k\to\infty} a_k = -1\,, \quad \text{and} \quad \lim_{k\to\infty} b_k = 1\,,$$

then, the equation $f(x) = 0$

- ☐ has an infinite amount of solutions
- ☐ has exactly one solution
- ☐ has exactly two solutions
- ☐ has no solution

Question 17 : Let $(a_n)_{n\geq 0}$ be a sequence such that $\lim\limits_{n\to\infty} a_n = 2$, and let $(b_n)_{n\geq 0}$ be the sequence given by

$$b_n = 1 + a_n \cos\left(n\frac{\pi}{2}\right)\,, \qquad n \geq 0\,.$$

Then

- ☐ $\liminf\limits_{n\to\infty} b_n = -1$ and $\limsup\limits_{n\to\infty} b_n = 3$
- ☐ $\liminf\limits_{n\to\infty} b_n = 3$ and $\limsup\limits_{n\to\infty} b_n = 3$
- ☐ $\liminf\limits_{n\to\infty} b_n = 1$ and $\limsup\limits_{n\to\infty} b_n = 3$
- ☐ $\liminf\limits_{n\to\infty} b_n = -2$ and $\limsup\limits_{n\to\infty} b_n = 2$

Question 18 : The integral $\displaystyle\int_0^1 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$ equals

- ☐ $\frac{8 - 2\sqrt{2}}{3}$
- ☐ $\frac{4 - 2\sqrt{2}}{3}$
- ☐ $4 - 2\sqrt{2}$
- ☐ $\frac{\sqrt{2} + 1}{3}$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $(a_n)_{n\geq 0}$ be a bounded sequence such that for all $n \in \mathbb{N}$, $a_n > 3$. Then $\liminf_{n\to\infty} a_n > 3$.

☐ TRUE ☐ FALSE

Question 20 : There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ which is both bounded and injective.

☐ TRUE ☐ FALSE

Question 21 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Then

$$\frac{1}{2}f(t)^2 = \frac{1}{2}f(0)^2 + \int_0^t f(x)f'(x)\,dx \qquad \forall t \in \mathbb{R}.$$

☐ TRUE ☐ FALSE

Question 22 : If the power series $\sum_{k=1}^{\infty} a_k (x-1)^k$ converges at $x = 0$, then it converges for $x = 2$.

☐ TRUE ☐ FALSE

Question 23 : Let f and g be two functions with taylor expansion of order 1 around $x_0 = 0$ given by

$$f(x) = 1 + 2x + x\varepsilon(x)\,,$$
$$g(x) = 1 + \frac{1}{2}x + x\varepsilon(x)\,.$$

Then, the Taylor expansion of order 1 of $f(g(x))$ around $x_0 = 0$ is given by

$$f(g(x)) = 3 + x + x\varepsilon(x)\,.$$

☐ TRUE ☐ FALSE

Question 24 : The roots of the polynomial $z^4 + z^3 - 2z^2 + 2z + 4$ are $\{-2, -1, \frac{1}{4}, 1+i\}$.

☐ TRUE ☐ FALSE

Question 25 : Let $f\colon\ ]0,1[\ \to \mathbb{R}$ be a continuous function. Then there exists a continuous function $g\colon [0,1] \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in\ ]0,1[$.

☐ TRUE ☐ FALSE

Question 26 : Let $A, B \subset \mathbb{R}$ be two bounded nonempty sets and $c \in \mathbb{R}$. Then,

$$\sup\{x + c : \ x \in A\} - \sup\{x + c : \ x \in B\} = \sup A - \sup B.$$

☐ TRUE ☐ FALSE

Question 27 : Let $f : \mathbb{R} \to]0, \infty[$ be a differentiable function. Then the function $g : \mathbb{R} \to]0, \infty[$ defined by $g(x) = f(x)^{f(x)}$ is also differentiable on $\mathbb{R}$.

☐ TRUE ☐ FALSE

Question 28 : Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences surch that for all $n \in \mathbb{N}$, $0 < a_n < b_n$. If the series $\sum_{n=0}^{\infty} a_n$ diverges, then the series $\sum_{n=0}^{\infty} \frac{1}{b_n}$ converges.

☐ TRUE ☐ FALSE

Third Part: open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 29: *This question is worth 7 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 *Do not write here.*

(a) Give the definition for $f\colon I \to \mathbb{R}$ to be uniformly continuous for interval I.

(b) Write, with justification an example of a continuous $f\colon \mathbb{R} \to \mathbb{R}$ but not uniformly continuous function.

(c) Does there exists a function $f\colon [-1,1] \to \mathbb{R}$ that is continuous but not uniformly continuous ? Justify.

Question 30: *This question is worth 4 points.*

☐0 ☐1 ☐2 ☐3 ☐4 *Do not write here.*

Show by induction or otherwise that

$$\forall n \in \mathbb{N}^*,\ \sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2).$$

Question 31: *This question is worth 5 points.*

0 1 2 3 4 5 *Do not write here.*

(a) Let $(a_n)_{n\geq 1}$ be a sequence and $l \in \mathbb{R}$. Give the definition of "The sequence $(a_n)_{n\geq 1}$ converges to l".

(b) Let $(x_n)_{n\geq 1}$ be a sequence such that both subsequences $(x_{2k})_{k\geq 1}$ and $(x_{2k+1})_{k\geq 1}$ converge to the same limit l. Show that $(x_n)_{n\geq 1}$ converges to l.