Rappel: Techniques d'intégration

Proposition. Formule de changement de variable.

$f : [a,b] \to \mathbb{R}$ continue, $\varphi : [\alpha,\beta] \to [a,b]$ continûment dérivable sur $I \supset [\alpha,\beta]$.

Alors
$$\boxed{\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\cdot\varphi'(t)\,dt}$$
où $x = \varphi(t)$

$d\,\varphi(t) = \varphi'(t)\,dt$

Ex 1.
$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{\overbrace{-2x\,dx}^{du}}{\underbrace{\sqrt{1-x^2}}_{\sqrt{u}}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}du = -\frac{1}{2}\,\frac{1}{\frac{1}{2}}\,u^{\frac{1}{2}} + C = -u^{\frac{1}{2}} + C =$$
$$= -\sqrt{1-x^2} + C$$

intégrale indéfinie = la plus générale primitive

$u = 1-x^2 \Rightarrow du = -2x\,dx$

Ex 2. $\int \frac{dx}{\sin x} = \int \frac{\overbrace{\sin x}^{-du}}{\underbrace{\sin^2 x}_{1-u^2}}\, dx = -\int \frac{du}{1-u^2} = -\int \frac{du}{(1-u)(1+u)} = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du =$

$u = \cos x \Rightarrow du = -\sin x\, dx$

$\sin^2 x = 1 - \cos^2 x$

$= -\frac{1}{2}\int \frac{du}{1-u} - \frac{1}{2}\int \frac{du}{1+u} = \frac{1}{2}\ln|1-u| - \frac{1}{2}\ln|1+u| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$

$\int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx = \frac{1}{2}\int \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\, dx + \frac{1}{2}\int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\, dx = \int \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\, d\left(\frac{x}{2}\right) + \int \frac{\cos\frac{x}{2}}{\sin\left(\frac{x}{2}\right)}\, d\left(\frac{x}{2}\right) =$

$= -\int \frac{d\left(\cos\frac{x}{2}\right)}{\cos\frac{x}{2}} + \int \frac{d\left(\sin\frac{x}{2}\right)}{\sin\frac{x}{2}} = -\ln\left|\cos\frac{x}{2}\right| + \ln\left|\sin\frac{x}{2}\right| + C$

$1 - \cos x = 2\sin^2\frac{x}{2}$

$1 + \cos x = 2\cos^2\frac{x}{2}$

$\left(\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha\end{aligned}\right)$

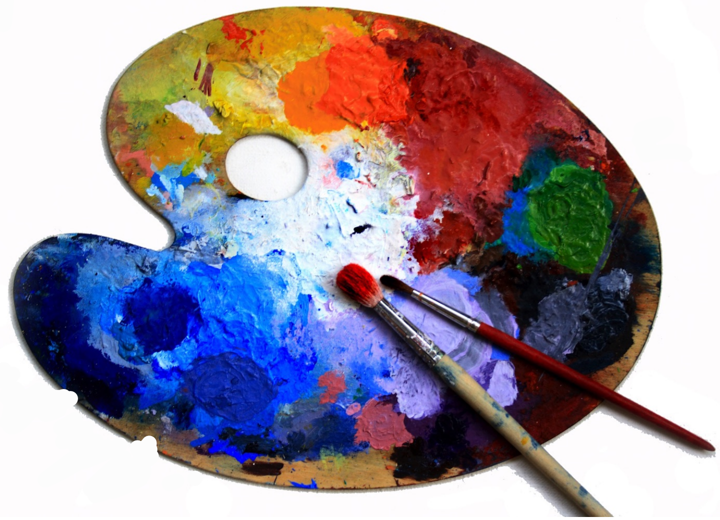
$\Rightarrow \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| = \frac{1}{2}\ln\left|\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\right| = \frac{1}{2}\ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right|^2 = \ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right| = \ln\left|\sin\frac{x}{2}\right| - \ln\left|\cos\frac{x}{2}\right|$

Remarque. Parfois deux expressions différentes sont égales à une constante près

$$\ln 5x + C_1 = \ln x + \ln 5 + C_2$$

$$\frac{1}{\cos^2 x} = \tan^2 x + C \quad : \quad \tan^2 x = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$$

Calcul des dérivée : =



Calcul des intégrales =

Proposition. **Formule d'intégration par parties.**

$g, f : I \to \mathbb{R}$ continûment dérivable ; $[a,b] \subset I$. Alors

$$\boxed{\int_a^b f(x)\, g'(x)\, dx = f(x)g(x)\Big|_a^b - \int_a^b g(x) f'(x)\, dx.}$$

Dém. $(f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x) \Rightarrow f(x)\cdot g'(x) = (f\cdot g)'(x) - f'(x)g(x)$

$$\Rightarrow \int_a^b f(x)\cdot g'(x)\, dx = \int_a^b (f\cdot g)'(x)\, dx - \int_a^b f'(x)\cdot g(x)\, dx = f(x)g(x)\Big|_a^b - \int_a^b g(x) f'(x)\, dx.$$

primitive de $(f\cdot g)'$

$\blacksquare$

$$\boxed{\int_a^b f\, dg = fg\Big|_a^b - \int_a^b g\, df}$$

Notations utiles:

$$d f(x) = f'(x)\, dx$$

$$f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a).$$

Ex 3.

$$\int_1^2 \underbrace{x^2}_{g'(x)} \underbrace{\ln x}_{f(x)}\, dx = \frac{1}{3}x^3 \ln x \Big|_1^2 - \int_1^2 \frac{1}{3}x^3 \cdot \frac{1}{x}\, dx = \frac{1}{3}x^3 \ln x \Big|_1^2 - \frac{1}{3}\int_1^2 x^2 dx =$$

$$g(x) = \frac{1}{3}x^3$$

$$= \frac{1}{3} 8 \ln 2 - \frac{1}{9} x^3 \Big|_1^2 = \frac{8}{3}\ln 2 - \frac{8}{9} + \frac{1}{9} = \frac{8}{3}\ln 2 - \frac{7}{9}$$

Marche bien pour $\int e^{ax}(\text{poly})$, $\int (\ln x)^k (\text{poly})$, $\int (\sin x, \cos x)(\text{poly})$, $\int (e^{ax})(\sin x\ \cos x)$.

Ex 4.

$$\int \sin x\, e^x\, dx$$

$$g'(x) = \sin(x)$$

$$g(x) = -\cos(x)$$

$$= \underbrace{-\cos(x)\, e^x}\Big| - \int -\cos(x)(e^x)'$$

$$= -\cos(x)\, e^x + \int \cos(x)\, e^x$$

$$\cancel{g'(x) = \cos(x)}$$

$$= \underbrace{-\cos(x)\, e^x + \sin(x)\, e^x}_{J(x)} - \underbrace{\int \sin(x)\, e^x}_{I}$$

$$2I = J$$

$$I = \frac{J}{2} + C =$$

$$= \frac{1}{2}(\sin x - \cos x)\, e^x + C$$

$$2I = J(x)$$

$$I = \frac{J(x)}{2} + C$$

Ex 5. $\int \underbrace{\ln(1+\sqrt{1+x})}_{f(x)} \underbrace{dx}_{g'(x)}$

$g(x) = x$ $\quad f(x) = \ln(1+\sqrt{1+x})$

$g'(x) = 1$ $\quad f'(x) = \dfrac{1}{2(1+\sqrt{1+x})\sqrt{1+x}}$

$$= x\ln(1+\sqrt{1+x}) - \int \frac{x}{2(1+\sqrt{1+x})\sqrt{1+x}}dx$$

$U = \sqrt{1+x} \iff x = U^2 - 1$

$dU = \dfrac{1}{2\sqrt{1+x}}$

$$= x\ln(1+\sqrt{1+x}) - \int \frac{U^2-1}{1+U}dU$$

$$= x\ln(1+\sqrt{1+x}) - \int \frac{\cancel{(1+U)}(U-1)}{\cancel{1+U}}du$$

$$= x\ln(1+\sqrt{1+x}) - \int (U-1)dU$$

$$= x\ln(1+\sqrt{1+x}) + U - \frac{U^2}{2} + C$$

$$= x\ln(1+\sqrt{1+x}) + \sqrt{1+x} - \frac{1+x}{2} + C$$

$$= x\ln(1+\sqrt{1+x}) + \sqrt{1+x} - \frac{1}{2}(1+x) + C$$

Intégration des fonctions rationnelles.

$\int \frac{P(x)}{Q(x)}\,dx$ s'exprime toujours en termes des fonctions élémentaires.

(Par contre, $\int \sin x \cdot \ln x \, dx$ ne s'exprime pas en termes des fonctions élémentaires).

poly

$$\frac{f(x)}{g(x)} = p(x) + \sum_i \frac{f_i(x)}{g_i(x)} \quad \text{où} \quad g_i(x) = \begin{cases} (ax+b)^k \Rightarrow f_i(x) = A_i \\ (ax^2+bx+c)^m \Rightarrow f_i(x) = A_i x + B_i \\ \text{si } b^2-4ac<0 \end{cases}$$

poly poly

(1) $$\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{du}{u} = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C$$

$a \neq 0$

$u = ax+b \Rightarrow du = a\,dx$

(2) $\displaystyle\int \frac{(cx+d)\,dx}{(x-a)(x-b)} = \int \left(\frac{A}{x-a} + \frac{B}{x-b}\right) dx = A \ln|x-a| + B \ln|x-b| + C$

$a \neq b$

Coefficients indéterminés $\displaystyle\frac{A}{x-a} + \frac{B}{x-b} = \frac{cx+d}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$

$$\Rightarrow \begin{array}{l} 1 \\ x \end{array}\begin{cases} -Ab - Ba = d \\ A + B = c \end{cases} \Rightarrow \begin{cases} B = c - A \\ -Ab - ac + Aa = d \end{cases} \Rightarrow \begin{array}{l} A = \dfrac{ac+d}{a-b} \quad a \neq b \\ B = c - A. \end{array}$$

(3) $\displaystyle\int \frac{dx}{(ax+b)^k} = \frac{1}{a}\int \frac{du}{u^k} = \frac{1}{a}\int u^{-k}\,du = \frac{1}{a}\,\frac{1}{-k+1}\,u^{-k+1} + C = \frac{1}{a(1-k)}(ax+b)^{-k+1} + C$

$k \geq 2$ $\quad u = ax+b$

$du = a\,dx$

(4) $\displaystyle\int \frac{dx}{x^2+c^2} = \frac{1}{c^2}\int \frac{dx}{\left(\frac{x}{c}\right)^2+1} = \frac{1}{c^2}\int \frac{c\,dt}{t^2+1} = \frac{1}{c}\arctan t + C = \frac{1}{c}\arctan\frac{x}{c} + C$

$t = \frac{x}{c} \Rightarrow dt = \frac{1}{c}dx$

$\displaystyle\left(\text{Aussi } \int \frac{dx}{x^2+px+q}\right) = \int \frac{dx}{\left(x+\frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)} = \int \frac{du}{u^2+c^2}$ $\quad u = \left(x+\frac{p}{2}\right),\ du = dx,\ c = \sqrt{\underbrace{q - \tfrac{p^2}{4}}_{>0}}$

$p^2 - 4q < 0$

(5) $\displaystyle\int \frac{x\,dx}{x^2+c^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+c^2| + C$

$u = x^2+c^2,\ du = 2x\,dx$

Ex. 6 $\displaystyle\int_0^1 \frac{dx}{x^2+x-6} = \int_0^1 \frac{dx}{(x-2)(x+3)} = \frac{1}{5}\int_0^1 \left(\frac{1}{x-2} - \frac{1}{x+3}\right)dx = \frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3| \Big|_0^1 =$

$$\frac{1}{(x-2)(x+3)} = \frac{1}{5}\left(\frac{1}{x-2} - \frac{1}{x+3}\right)$$

$$= \frac{1}{5}(\ln 1 - \ln 2) - \frac{1}{5}(\underbrace{\ln 4}_{2\ln 2} - \ln 3) =$$

$$= \frac{1}{5}(-3\ln 2 + \ln 3) = \frac{1}{5}\ln\frac{3}{8}$$

Ex. 7. $\displaystyle\int_{-2}^0 \frac{dx}{x^2+4x+8} = \int_{-2}^0 \frac{dx}{(x^2+4x+4)+4} = \int_{-2}^0 \frac{dx}{(x+2)^2+4} = \int_0^2 \frac{du}{u^2+4} =$

$u = x+2$, $x=-2 \Rightarrow u=0$
$du = dx$, $x=0 \Rightarrow u=2$

$$= \frac{1}{4}\int_0^2 \frac{du}{\left(\frac{u}{2}\right)^2+1} = \frac{1}{2}\int_0^1 \frac{dt}{t^2+1}$$

$$\frac{1}{2}\arctan t \Big|_0^1 = \frac{1}{2}\cdot\frac{\pi}{4} - 0 = \frac{\pi}{8}.$$

$t = \frac{u}{2} \Rightarrow dt = \frac{1}{2}du$
$u = 0 \Rightarrow t = 0$
$u = 2 \Rightarrow t = 1$

Question 25

L'integrale $\int_0^1 x^2 e^{2x}\,dx$ vaut

$df(x) = f'(x)\,dx$

A. $e^2 - \frac{1}{2}$

B. $\frac{3}{4}e^2 - \frac{1}{4}$

C. $\frac{1}{4}e^2 - \frac{1}{4}$

D. $\frac{1}{2}e^2 - \frac{1}{2}$

E. $\frac{5}{4}e^2 - \frac{1}{4}$

$$\int x^2 e^{2x}\,dx = \frac{1}{2}\int \underbrace{x^2}_{f}\,\underbrace{d(e^{2x})}_{dg} = \underbrace{\frac{1}{2}x^2 e^{2x}}_{f\cdot g} - \frac{1}{2}\int \underbrace{e^{2x}}_{g}\cdot\underbrace{2x\,dx}_{df} =$$

$$= \frac{1}{2}x^2 e^{2x} - \int e^{2x}x\,dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}\int \underbrace{x}_{f}\,\underbrace{d\,e^{2x}}_{dg} =$$

$$= \frac{1}{2}x^2 e^{2x} - \underbrace{\frac{1}{2}x e^{2x}}_{f\cdot g} + \frac{1}{2}\int \underbrace{e^{2x}}_{g}\,\underbrace{dx}_{df} = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x}\Big|_0^1 =$$

$$= \frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{4}e^2 - \frac{1}{4} = \frac{1}{4}e^2 - \frac{1}{4}$$