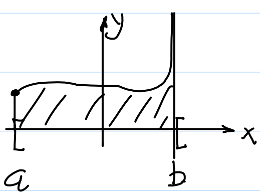
Chapitre 12

Intégration (chapitres choisis)

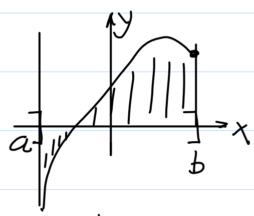
12.4. Intégrales généralisées (ou impropres)

12.4.1. Exposition des trois types

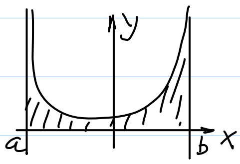
i) type 1 : f continue sur $[a,b[$ ou $]a,b]$ ou $]a,b[$.
$a, b \in \mathbb{R}$, $a < b$.



$$\int_a^{b-} f(x)\,dx$$

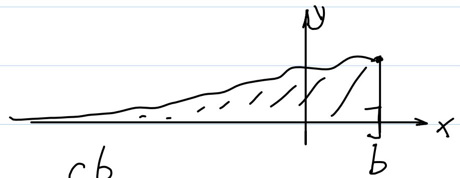


$$\int_{a+}^{b} f(x)\,dx$$

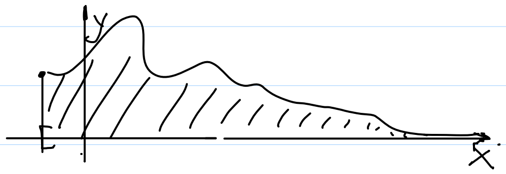


$$\int_{a+}^{b-} f(x)\,dx.$$

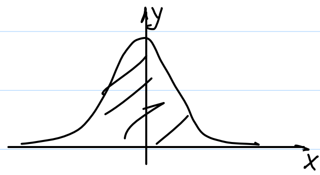
ii) type 2 : f continue sur $]-\infty, b]$, $[a,+\infty[$, $]-\infty,+\infty[$



$$\int_{-\infty}^{b} f(x)\,dx$$

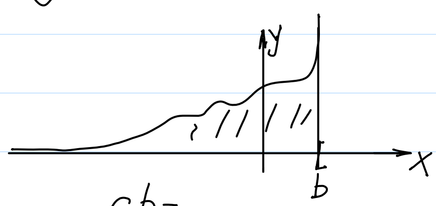


$$\int_a^{\infty} f(x)\,dx$$

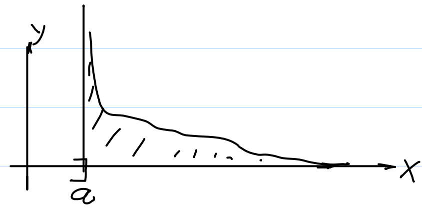


$$\int_{-\infty}^{\infty} f(x)\,dx.$$

iii) type 3 combinaison de i) et ii)

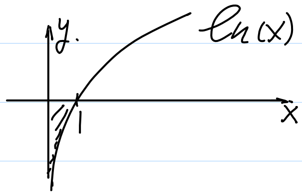


$$\int_{-\infty}^{b-} f(x)\,dx$$

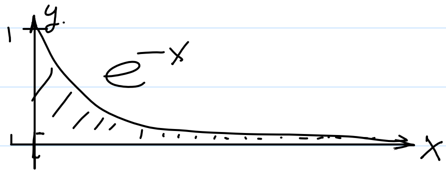


$$\int_{a+}^{+\infty} f(x)\,dx$$

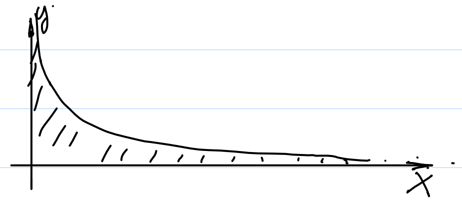
Exemples explicites



$$\int_{0+}^{1} \ln(x)\, dx = ? \qquad \text{type 1}$$



$$\int_{0}^{+\infty} e^{-x}\, dx = ? \qquad \text{type 2}$$



$$\int_{0+}^{+\infty} e^{-x} \frac{1}{\sqrt{x}}\, dx = ? \qquad \text{type 3}$$

12.4.2. Intégrales du type 1

Définition (type 1)

- Si f est continue sur $[a, b[$ — f continue sur $[a,\beta]$

$$\int_{a}^{b-} f(x)\, dx := \lim_{\beta \to b-} \int_{a}^{\beta} f(x)\, dx \quad , \; a < \beta < b$$

- Si f est continue sur $]a, b]$ — f continue sur $[\alpha, b]$

$$\int_{a+}^{b} f(x)\, dx := \lim_{\alpha \to a+} \int_{\alpha}^{b} f(x)\, dx \quad , \; a < \alpha < b$$

- Si f est continue sur $]a, b[$ — $\begin{cases} a < \alpha < \beta < b \\ f \text{ continue sur } [\alpha, \beta] \end{cases}$

$$\int_{a+}^{b-} f(x)\, dx := \lim_{\substack{\alpha \to a+ \\ \beta \to b-}} \int_{\alpha}^{\beta} f(x)\, dx$$

$\Big\{$ une limite après l'autre, l'ordre est irrelevant.

12.4.3. Intégrales du type 2

Définition (type 2)

- Si f est continue sur $[a, +\infty[$, alors

$$\int_a^{+\infty} f(x)\,dx := \lim_{R \to \infty} \int_a^R f(x)\,dx \quad , \quad a < R$$

f continue sur $[a,R]$

- Si f est continue sur $]-\infty, b]$, alors

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{R \to -\infty} \int_R^b f(x)\,dx \quad , \quad R < b$$

f continue sur $[R,b]$

- Si f est continue sur $]-\infty, +\infty[\equiv \mathbb{R}$, $R_1 < R_2$.

$$\int_{-\infty}^{+\infty} f(x)\,dx := \lim_{\substack{R_1 \to -\infty \\ R_2 \to +\infty}} \int_{R_1}^{R_2} f(x)\,dx$$

une limite après l'autre, l'ordre est irrelevant.

12.4.4. Intégrales du type 3

Définition (type 3)

Si f est continue sur $]a, +\infty[$ ou $]-\infty, b[$, alors

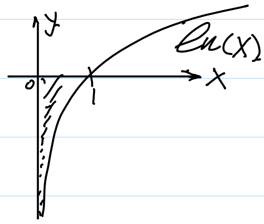
$$\int_{a+}^{+\infty} f(x)\,dx := \lim_{\substack{\alpha \to a+ \\ R \to +\infty}} \int_\alpha^R f(x)\,dx \quad , \quad a < \alpha < R < +\infty$$

$$\int_{-\infty}^{b-} f(x)\,dx := \lim_{\substack{\beta \to b- \\ R \to -\infty}} \int_R^\beta f(x)\,dx \quad , \quad -\infty < R < \beta < b$$

une limite après l'autre, l'ordre est irrelevant.

12.5. Exemples d'intégrales généralisées

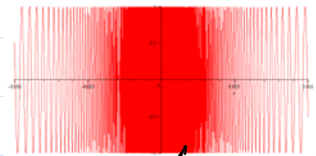
12.5.1 Exemples (type 1)



1) $$\int_{0+}^{1} \ln(x)\,dx := \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} 1\cdot \ln(x)\,dx =$$

$$= \lim_{\varepsilon\to 0+} \left(\left[x\cdot \ln(x) \right]_{\varepsilon}^{1} - \int_{\varepsilon}^{1} x\cdot \frac{1}{x}\,dx \right) =$$

$$= \lim_{\varepsilon\to 0+} \left(0 - \varepsilon\cdot \ln(\varepsilon) - \left[x \right]_{\varepsilon}^{1} \right) = \lim_{\varepsilon\to 0+} (-1+\varepsilon) = -1.$$



2) $$\int_{0+}^{1} \sin\left(\frac{1}{x}\right) dx := \lim_{\varepsilon\to 0+} \int_{\varepsilon}^{1} \underbrace{\sin\left(\frac{1}{x}\right)\frac{1}{x^2}}\, x^2\,dx =$$

$$= \lim_{\varepsilon\to 0+} \left(\left[\cos\left(\frac{1}{x}\right) x^2 \right]_{\varepsilon}^{1} - 2\cdot \int_{\varepsilon}^{1} \underbrace{\cos\left(\frac{1}{x}\right)\cdot x}_{\text{continue sur } [0,1] \text{ par prolongement continu}}\,dx \right)$$

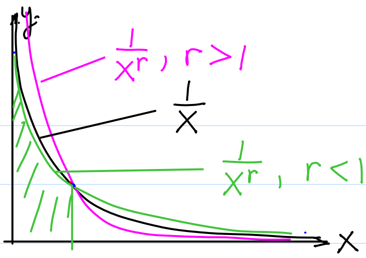
$$= \cos(1) - 2\int_{0}^{1} \underbrace{\cos\left(\frac{1}{x}\right)\cdot x}_{\text{pas de fonction élémentaire comme primitive mais l'intégrale est bien définie}}\,dx$$

⌜car $\lim_{\varepsilon\to 0+} \int_{\varepsilon}^{1} f(x)\,dx = \int_{0}^{1} f(x)\,dx$ pour f une fonction continue sur $[0,1]$ ⌟

12.5.2. Les fonctions $\frac{1}{x^r}$, $r>0$, sur $]0,1]$ (type 1)

Pour $r>0$, $r\neq 1$ on a:

$$\int_{0+}^{1} \frac{1}{x^r}\,dx = \lim_{\varepsilon\to 0+} \int_{\varepsilon}^{1} \frac{1}{x^r}\,dx = \lim_{\varepsilon\to 0+} \left[\frac{1}{1-r}\,\frac{1}{x^{r-1}} \right]_{\varepsilon}^{1} =$$



$$= \lim_{\varepsilon \to 0+} \left(\frac{1}{1-r} - \frac{1}{1-r}\,\frac{1}{\varepsilon^{r-1}} \right) = \begin{cases} +\infty & \text{si } r>1 \\ \dfrac{1}{1-r} & \text{si } r<1. \end{cases}$$

et pour $r=1$ on trouve:

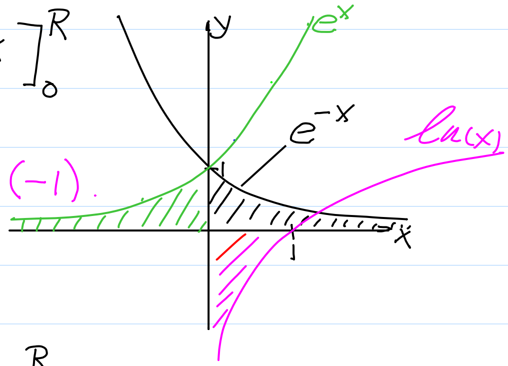
$$\int_{0+}^{1} \frac{1}{x}\,dx = \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{1}{x}\,dx = \lim_{\varepsilon \to 0+} \left[\ln(x) \right]_{\varepsilon}^{1} =$$

$$= \lim_{\varepsilon \to 0+} \left(-\ln(\varepsilon) \right) = +\infty$$

12.5.3 Exemples (type 2)

1) $$\int_0^{+\infty} e^{-x}\,dx = \lim_{R \to +\infty} \int_0^R e^{-x}\,dx = \lim_{R \to +\infty} \left[-e^{-x} \right]_0^R$$

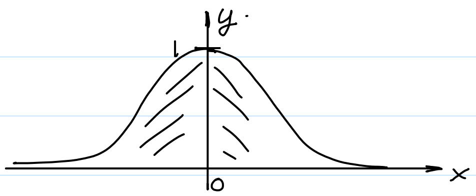
$$= \lim_{R \to +\infty} \left(-e^{-R} + e^{-0} \right) = 1 = 1 = -(-1).$$

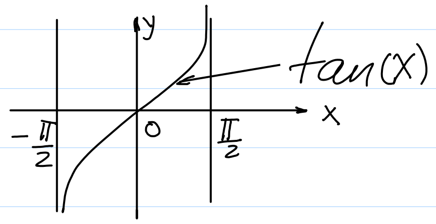


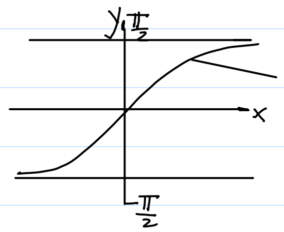
2) $$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = 2 \cdot \int_0^{+\infty} \frac{1}{1+x^2}\,dx = 2 \cdot \lim_{R \to +\infty} \int_0^R \frac{1}{1+x^2}\,dx =$$

$$= 2 \cdot \lim_{R \to \infty} \left[\arctan(x) \right]_0^R =$$

$$= 2 \cdot \lim_{R \to \infty} \arctan(R) = \pi.$$





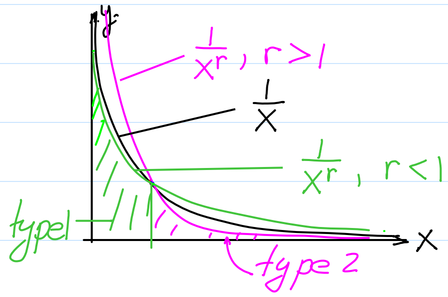


arctan(x)

12.5.4. Les fonctions $\frac{1}{x^r}$, $r>0$, sur $[1,+\infty[$ (type 2)

Pour $r>0$, $r\neq 1$ on a:

$$\int_1^{+\infty}\frac{1}{x^r}\,dx = \lim_{R\to+\infty}\int_1^R \frac{1}{x^r}\,dx = \lim_{R\to+\infty}\left[\frac{1}{1-r}\,\frac{1}{x^{r-1}}\right]_1^R =$$



$$=\lim_{R\to+\infty}\left(\frac{1}{1-r}\,\frac{1}{R^{r-1}}-\frac{1}{1-r}\right) = \begin{cases}+\infty & \text{si } r<1\\ \frac{1}{r-1} & \text{si } r>1\end{cases}$$

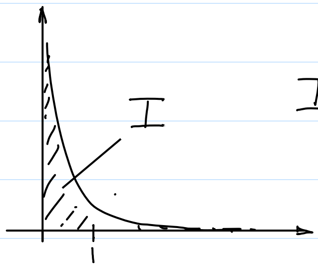
et pour $r=1$ on trouve:

$$\int_1^{+\infty}\frac{1}{x}\,dx = \lim_{R\to+\infty}\int_1^R\frac{1}{x}\,dx = \lim_{R\to+\infty}\left[\ln(x)\right]_1^R =$$

$$= \lim_{R\to+\infty}\ln(R) = +\infty.$$

2.5.5 Exemple (type 3)

$$\int_{0+}^{+\infty}\frac{1}{\sqrt{x}}\,e^{-x}\,dx = \lim_{\substack{R\to+\infty\\ \varepsilon\to 0+}}\int_\varepsilon^R\frac{1}{\sqrt{x}}\,e^{-x}\,dx =: I$$



$$I = \lim_{\substack{R\to+\infty\\ \varepsilon\to 0+}}\int_{\sqrt{\varepsilon}}^{\sqrt{R}}\underbrace{\frac{1}{u}\,e^{-u^2}\overbrace{(2u)}^{\varphi'(u)}}_{\text{continue en } u=0}\,du = \quad !$$

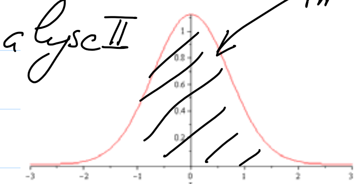
changement de variables

$x=\varphi(u)=u^2$, $\varphi(\sqrt{\varepsilon})=\varepsilon$, $\varphi(\sqrt{R})=R$

$$= \lim_{R\to+\infty} 2\int_0^R e^{-u^2}\,du = \sqrt{\pi}$$

voir Analyse II

$\frac{1}{\sqrt{\pi}}e^{-u^2}$



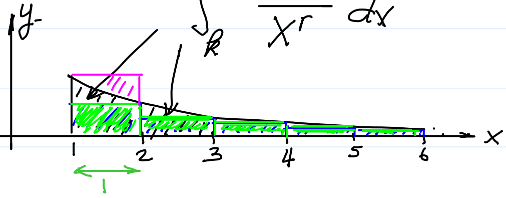
$$\Longrightarrow \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\,e^{-u^2}\,du = 1$$

12.6. Convergence de séries numériques

1)

$$\int_k^{k+1} \frac{1}{x^r}\,dx$$

$$\underline{r>1:}\quad \underbrace{\int_1^{+\infty} \frac{1}{x^r}\,dx}_{=\frac{1}{r-1}} = \lim_{n\to\infty} \int_1^{n+1} \frac{1}{x^r}\,dx =$$



$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x^r}\,dx = \sum_{k=1}^{\infty} \int_k^{k+1} \frac{1}{x^r}\,dx \geqslant$$

$$\geqslant \sum_{k=1}^{\infty} \frac{1}{(k+1)^r}\cdot 1 = \sum_{k=2}^{\infty} \frac{1}{k^r} \overset{-1+1=0}{=} \underline{-1 + \sum_{k=1}^{\infty} \frac{1}{k^r}}$$

$$\Longrightarrow \quad \boxed{\sum_{k=1}^{\infty} \frac{1}{k^r} \leqslant 1 + \frac{1}{r-1} = \frac{r}{r-1}\,, \quad r>1}$$

2) $$\sum_{k=2}^{\infty} \frac{1}{k\cdot \ln(k)} \geqslant \lim_{n\to\infty} \int_2^{n} \frac{1}{x\ln(x)}\,dx$$

$$= \lim_{n\to\infty} \Big[\ln(\ln(x))\Big]_{x=2}^{x=n} = \infty.$$

<u>donc cette série diverge.</u>

3) par contre on a pour $p>1$ que

$$\sum_{k=2}^{\infty} \frac{1}{k\,\ln(k)^p} < \infty \qquad (\text{le montrer !})$$

12.7. Intégration des fonctions rationnelles

1) Décomposition de $q(x)$ en facteurs réels irréductibles

Exemples:

$$x^2-1 = (x-1)(x+1)$$
$$x^2+1$$
$$x^3-1 = (x-1)(x^2+x+1)$$
$$x^3+1 = (x+1)(x^2-x+1)$$
$$x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$
$$x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$$

voir le chapitre des nombres complexes

2) Décomposition de $f(x) = \frac{p(x)}{q(x)}$ en éléments simples

Exemple: $f(x) = \dfrac{2x^3}{x^4-1} = \dfrac{\alpha}{x-1} + \dfrac{\beta}{x+1} + \dfrac{\gamma x+\delta}{x^2+1}.$

$$\Rightarrow 2x^3 = \alpha(x+1)(x^2+1) + \beta(x-1)(x^2+1) + (\gamma x+\delta)(x^2-1)$$

$$\left.\begin{array}{ll} x^3: & 2 = \alpha + \beta + \gamma \\ x^2: & 0 = \alpha - \beta + \delta \\ x: & 0 = \alpha + \beta - \gamma \\ 1: & 0 = \alpha - \beta - \delta \end{array}\right\}$$

algèbre linéaire

$\alpha = \beta = \frac{1}{2}, \ \gamma = 1, \ \delta = 0$