Assignment 3

1. Kyle (1985) with activism

Consider a Kyle (1985) model where the risk-neutral 'activist' is endowed at date 0 with $X_0 \sim N(\mu_x, \sigma_x^2)$ shares in a firm whose terminal payoff is v can be chosen by the activist at time $T = 1$ by paying a cost equal to $C(v) = \frac{c}{2}v^2$. The initial position X_0 are known to the 'insider' at date 0 and will become public at date $T = 1$ to all market participants after the activist announces her choice. Suppose that the activist can buy additional shares in the market at date 0 by sending market orders for $\Delta X = X_1 - X_0$ shares to the market-maker who will execute the trade at price P_1. The market maker sets the price P_1 competitively, so as to have zero expected profit across all her trades. In addition to orders from the insider, the market maker receives market orders from noise traders $U \sim N(0, \sigma_u^2)$. Thus the market maker observes only the total order flow $Y = \Delta X + U$. Assume that investors can also invest at a risk-free rate that we set to zero for simplicity.

1. Solve for the equilibrium level of activism effort, the trading strategy, and the equilibrium price. Does an equilibrium always exist? Is it unique?
2. How does a change in the parameters c, σ_X, σ_u change (i) the market liquidity, i.e., Kyle's lambda, (ii) the equilibrium price, (iii) the expected profit of the activist? Can you interpret?

2. Dynamic Grossman-Stiglitz (based on Watanabe (1998))

Assume an economy is populated of N myopic CARA agents who each live for 2 periods, consume in the first period and hand over their second period wealth to their direct descendants, so each time t generation maximizes $E_t[e^{-\gamma C_t} + e^{-\rho} e^{-\gamma W_{t+1}}]$. Suppose they can invest in a risk-free asset with constant risk-free rate $R = 1 + r$ and in a risky stock with price S_t that pays a dividend D_t. Assume that the total number of shares outstanding is θ_t and that the joint dynamics of the state variables are given by:

$$D_{t+1} = D_t + \mu + x_t + \sigma_D \epsilon_{D,t+1} \quad (1)$$

$$x_{t+1} = \phi_x x_t + \sigma_x \epsilon_{x,t+1} \quad (2)$$

$$\theta_{t+1} = \phi_\theta \theta_t + \sigma_\theta \epsilon_{\theta,t+1} \quad (3)$$

where $\epsilon_D, \epsilon_x, \epsilon_\theta$ are independent standard normal random variables.

1. Assume that the time varying expected dividend component x_t is observable by all agents. Solve for the equilibrium price, consumption, and trading strategy of the agents assuming that they behave competitively with respect to prices. To that effect:
   - Conjecture that the price can be written as $P_t = \pi_0 + \pi_D D_t + \pi_x x_t + \pi_\theta \theta_t$.
   - Given this conjecture solve for the partial equilibrium optimal consumption (C_t), and trading strategy (n_t) of an agent of generation t, whose wealth W_t satisfies the dynamics:

     $$W_{t+1} = R(W_t - C_t) + n_t(P_{t+1} + D_{t+1} - RP_t).$$

     Show that it is of the form:

     $$C_t = aW_t + b(D_t, x, \theta_t) \quad (4)$$

     $$n_t = n(D_t, x_t, \theta_t) \quad (5)$$

     for some constant a, and some affine functions b, n for you to determine.
   - Imposing market clearing, i.e.,

     $$N * n(x_t, \theta_t) = \theta_t$$

     solve for all the equilibrium price coefficients.
2. Give an interpretation for the signs of π_D, π_X, π_θ. In particular, show that there exists two possible equilibria, characterized by different signs for π_θ. What happens to the stock price volatility when both $\sigma_D, \sigma_X \to 0$ in both equilibria? Which one seems more reasonable to you? Can you give some interpretation?
3. **Optional:** A few questions for you to think about.

- Can you solve the model if instead of being exogenous, you solve for the endogenous risk-free rate r that clears market for risk-free borrowing and lending? (how would you do it? would the risk-free rate be constant? Would there still be multiple equilibria? Check out the great paper by Loewenstein and Willard (JOF 2006 "the limits of investor behavior").
- Can you solve the model (with an exogenous risk-free rate) if x_t is not observable, but instead agents have to learn it from the observation of the dividend process? Now agents would have to learn $\hat{x}_t = E_t[x_t \mid F_t^D]$. What do you expect would change in the model?
- Suppose now that N_I agents observe x_t and N_u agents only observe D_t and the market clearing price P_t. Could you solve the model in that case? What do you expect would change? (Check out Wang (1994) and Watanabe (2008) for answers to the latter 2 questions).

3. Dynamic Speculation in Harison and Kreps (1978)

Consider the model of HK with two types of risk-neutral agents $i = \{a, b\}$ who trade one stock which pays a dividend $d_t \in \{1, 2\}$ in state $s_t \in \{1, 2\}$, which follows a discrete time Markov chain. The transition probability Matrix perceived by agent a is Q^a with transition probability $q^a_{ss'}$. Similarly for agent b but with different beliefs:

- for agent a: $q^a_{21} = 1/2$ $q^a_{12} = 2/3$
- for agent b: $q^b_{21} = 1/3$ $q^b_{12} = 1/4$

Both agents know each others' beliefs and agree to disagree! There are no short-sales. The risk-free rate R_f, discount rate $\gamma = 1/R_f = 0.75$.

1. Find v^a_1, the value of the strategy for agent a of buying and holding the stock forever when in state 1. Similarly derive v^a_2, v^b_1, v^b_2.
2. Can $p_1 = \max[v^a_1, v^b_1]$ and $p_2 = \max[v^a_2, v^b_2]$ be an equilibrium price system (i.e., where the equilibrium price $p(s_t) = \sum_s p_s \mathbf{1}_{s_t=s}$)? Give an argument to prove this is not an equilibrium.
3. Solve for the equilibrium price system following the argument of HK 1978.
4. Is it always true that when agents agree to disagree (i.e., when $q^a_{ss'} \neq q^b_{ss'}$) that the maximum of the buy and hold values cannot be an equilibrium price system?
5. **Optional:** A few questions to think about:
   - What would change if agents could learn about the true transition probabilities?
   - What would change if agents were risk-averse?
   - What would change if agents were allowed to short-sell (but possibly with a risk-limit)?
   - What would change if the beliefs of the other agents were not common-knowledge?