EPFL & SFI

**Information and Asset Prices**

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# Assignment 1

## 1. No Trade Theorem

Consider an economy with two dates 0 and 1. At date 1 there are two states $s = (a, b)$. There are two securities traded. One is a stock with time 0 price $S$ with payoff at date 1 of $D = (3, 6)$. The other is a risk-free bond with time 0 price $B$ and which pays 1 in every future state. All securities are in zero net supply in equilibrium. There are two agents $i = J, K$ who maximize their expected utility of consumption of the form $\log(c_0) + E[\log(c_1(s))]$ by trading in the two securities with each other at date 0.

Both agents have an initial endowment of $e_J(0) = 2$ and $e_K(0) = 1$. Their time 1 endowment is respectively $e_J(1) = (1, 2)$ and $e_K(1) = (2, 4)$.

We assume that it is common knowledge that at date $\epsilon \in (0, 1)$ (close to 0) agent $J$ will observe the realization of a (fair, i.e., 50/50) coin flip that reveals that $p_a = 0.75$ if tails occurs or $p_a = 0.25$ if heads occurs.

1. Derive the rational expectation equilibrium (i.e, the optimal consumption allocations, the price of the stock and the risk-free rate) in this economy at date 0 before the coin is flipped, that is when both agents have identical information.
2. Give state prices and risk-neutral probabilities for this economy. Are these unique? Is this economy arbitrage-free? Is this a complete market economy? Is the equilibrium Pareto Optimal? (justify each of your answer).
3. Assume that after trading has taken place at date 0 and agents have achieved a Pareto optimal allocation, the coin is flipped and agent J gets to observe the realization of the flip, but agent K does not. Show that even if agents were allowed to reopen markets for trading at date $\epsilon$, there is an equilibrium where the price of the stock and bond (and of Arrow Debreu securities) adjust, so that the equilibrium is fully revealing (i.e., agent K can infer the probability $p_a$ from the equilibrium arrow-debreu prices) and there is no trading in any securities. That is the initial Pareto optimal allocation remains pareto optimal.

   Derive the equilibrium prices of Arrow Debreu securities and of the stock and bond at date $\epsilon$, conditional on the coin draw.

## 2. Adding noise to the Grossman (1976) model

Consider an economy with $N$ agents with wealth $W_{i0}$ who can invest in risk-free asset with gross return $R_0$ and risky asset with price $P_0$ which pays off $v \sim N(\mu_v, \sigma_v^2)$ at time 1.

Each agent $i$ observes a **private signal** $y_i = v + \epsilon_i$. $\epsilon_i \sim N(0, \sigma_\epsilon^2)$ iid with $Cov(\epsilon_i, \epsilon_j) = 0$ and a **public signal** $P_0$. Agents have CARA expected utility with **same** risk-aversion $a$. Assume that the total supply of the risky asset is random, the number of shares is $X \sim N(\bar{X}, \sigma_X^2)$ (note that this contrasts from the Grossman (1976) model where supply is known and fixed). $X$ is independent of all individual signals $y_i$.

1. Prove the following result which should help you with the solution of the model. Suppose that $s_i = v + \epsilon_i \ \forall i = 1, \ldots, n$ with $v \sim N(\mu_v, \sigma_v^2)$ and $\epsilon_i \sim N(0, \sigma_i^2)$ and all random variables are independent. Then show that $E[v \,|\, s_1, \ldots, s_n] = E[v \,|\, \bar{S} = \sum_i \omega_i s_i]$ where $\omega_i = \frac{\tau_i}{\sum_j \tau_j}$ where we define the precision $\tau_i = 1/\sigma_i^2$. That is, $\bar{S}$ is a sufficient statistic for $v$ given all the signals. In turn, show that this implies that

$$E[v \,|\, s_1, \ldots, s_n] = \mu_v + \beta(\bar{S} - \mu_v) \tag{1}$$

$$\beta = \frac{\sum_j \tau_j}{\tau_v + \sum_j \tau_j} \tag{2}$$

$$Var[v \,|\, s_1, \ldots, s_n] = \sigma_v^2(1 - \beta) \tag{3}$$

2. Solve for the equilibrium price and the optimal demand by each agent. *hint: Notice that each agent will condition her demand on her private signal $y_i$ and a composite signal she can extract from the equilibrium price that will reflect a weighted average of the other agents signals and the noisy supply, similar to the approach in Grossman and Stiglitz (1981)).*

3. Discuss properties of the equilibrium and in particular how it differs from the Grossman (1976) model you should recover when $\sigma_x \to 0$. Is the equilibrium fully revealing? Does an individual's demand depend on the price? Does an equilibrium always exist?

4. Do you think the results extend in a straightforward fashion to the case where agents have different risk-aversion coefficients $a_i \ \forall i$?