The Adams-Sato effective conductivity model for snow

a) Context

The derivation of a physically based model for the thermal conductivity of snow as shown here serves as one important example of how we can approach the complex microstructure of snow. It relies on many simplifications but represents some of the main features of heat transport in snow. In particular, it has a physical representation of heat transport by small scale water vapour transport expressed in a way that it can simply be incorporated in the heat transfer equation without explicitly modelling vapour fluxes.
Recall that we would like to solve a one-dimensional heat transfer equation for snow in order to estimate the temperature gradient, which in turn serves as a basis for other process descriptions such as snow metamorphism. For snow, an adequate heat transfer equation is given by:

$$\rho_s c_p \frac{\partial T_s}{\partial t} - \frac{\partial}{\partial z}(k_h \frac{\partial T_s}{\partial z}) = Q_{pc} + Q_{sw} . \qquad \text{(1D-HeatTransfer)}$$

Here, the heat capacity for a unit volume of snow of a given density is calculated as:

$$\rho_s c_p = \rho_i c_i \theta_i + \rho_w c_w \theta_w + \rho_a c_a \theta_a \qquad \text{(HeatCapacity)}$$

The equation "1D-HeatTransfer" says that the local rate of change in snow temperature $T_s\ (K)$ is determined by temperature diffusion and two possible source terms, the source through the phase change processes melting / freezing or sublimation / inverse sublimation $Q_{pc}\ (\frac{W}{m^3})$ and the source through absorption of short wave radiation $Q_{sw}\ (\frac{W}{m^3})$, which is also a volume source for snow as explained in the section on the surface energy balance. The equation further needs the mean bulk snow density per layer $\rho_s\ (\frac{kg}{m^3})$ and the mean specific heat capacity $c_p (\frac{J}{kgK})$ for the mixture. These bulk properties are defined in Equation "Heat Capacity" by the mixture of the snow constituents ice, water and air expressed by their respective volume fractions $\theta_{i,w,a}\ (1)$. The parameter in question, the thermal conductivity $k_h\ (\frac{W}{mK})$ is a measure of the efficiency with which heat can diffuse through the snow.

b) Set-up of problem

Consider now as a simplification for now that snow is a composite of pure ice and pore space and constructed in a series and parallel arrangement of cubes such as in Figure "HeatFlux" In analogy to electricity resistance – current relationships or to the discussion of models for snow mechanics discussed in Chapter 3, you can call this the Burger's model of conductivity.
If we now consider heat flux through this snow cube, we can write the Fourier law of heat conduction for this small but finite hypothetical snow cube as:

$$q = -\frac{A}{l} k_h \Delta T . \qquad \text{(HeatFirstPriciple)}$$

Here $A\,(m^2)$ is the cross section of our unit snow block, $l\ (m)$ is its length, $k_h\ (\frac{W}{mK})$ the thermal conductivity of interest and $\Delta T\ (K)$ the bulk temperature difference over the block. We can define the total "resistance", $R_t\ (\frac{K}{W})$ to the heat transport therefore as:

$$R_t = \frac{l}{k_h A}, \qquad \text{(ThermalResistance)}$$

which can be represented by the individual parallel resistances:

$$\frac{1}{R_t} = \frac{1}{R_i} + \frac{1}{R_p} + \frac{1}{R_{ip}}, \qquad \text{(ParallelResistances)}$$

c) Conductivity in tortuous ice matrix

For the ice phase we now assume that we have a packing of spheres with contact areas $r_c\ (m)$ as illustrated in Figure "HeatFlux". Carslaw and Jaeger (1980) show that the resistance of a single grain to grain transition is given by:

$R_\propto = \frac{8}{3\pi^2 r_c k_i}$, where k_i is the thermal conductivity of pure ice. From geometrical considerations in a unit cell and pointing out that bonds between grains may have an isotropic distribution, Adams and Sato suggest the following relationship for the conductivity in the ice phase of snow:

$\frac{1}{R_i} = \frac{\pi^2 r_c k_i N}{32}$, where $N\ (1)$ is a dimensionless coordination number, i.e. the average number of connecting neighboring grains per grain. It is important to note that the sphere packing in the ice phase can be seen as a formal consideration of the tortuosity in the ice matrix of snow. In the ice matrix of any snow volume, heat cannot flow directly along the bulk temperature gradient, since the ice matrix does not always provide a direct connection along the gradient. This is expressed by the material tortuosity and assuming a sphere packing in our "ice column" is a (crude) approximation for this situation.

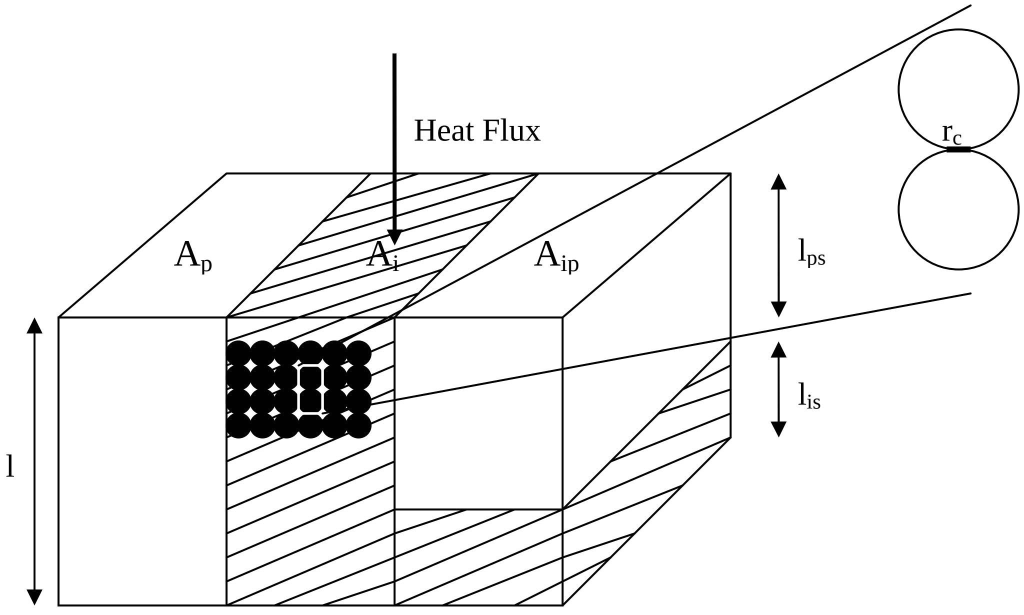

Figure “HeatFlux”: Schematic unit snow block for derivation of thermal conductivity. Terms are defined in the text.

d) Formal parameterization of vapour transport contribution

Now we have a formal representation of heat flow resistance or conductivity in the ice matrix of snow and it seems that the heat flow through the pore space is quite straight forward, namely given by the resistance of bulk air, $R_a = \frac{l}{k_a A}$. The complication, which we want to address, is water vapour transport, however. Warmer or more convex ice surfaces “eject” excess water molecules constantly into the pore space and at the same time there is a net deposition of vapour molecules at the colder or more concave surfaces. This process takes away energy from the warmer surfaces and locally adds heat to the colder surfaces resulting in a net transport of heat by water vapour diffusion through air expressed as a net vapour flux $F_v \ (\frac{kg}{m^2 s})$ across the pore space driven by the partial pressure of water vapour $p_v \ (\frac{N}{m^2})$:

$$F_v = -D_p \frac{\partial p_v}{\partial z}. \qquad \text{(VapourFlux)}$$

The diffusion coefficient of water vapour in snow based on a local pressure gradient is an inverse velocity $D_p \ (\frac{s}{m})$ and via simple expansion we can write:

$$F_v = -D_p \frac{\partial p_v}{\partial T} \frac{\partial T}{\partial z}. \qquad \text{(VapourFluxII)}$$

Now we take the divergence of this flux to get the local rate of change of mass:

$$\frac{\partial F_v}{\partial z} = \frac{\partial m}{\partial t} = \frac{Q_{pc}}{L} = -D_p \frac{\partial p_v}{\partial T}\frac{\partial^2 T}{\partial z^2}. \quad \text{(ChangeIceMass)}$$

Here we have assumed that the diffusion coefficient is independent of depth. The local ice mass $m\,(kg)$ changes as a result of a divergent vapour flux. This can also be expressed as a divergent heat flux $Q_{pc}(\frac{W}{m^3})$ divided by the latent heat of vaporization $L(\frac{J}{kg})$. If we regard now only the local (layer) situation, we can also re-write Equation "1D-HeatTransfer":

$$\rho_s c_p \frac{\partial T_s}{\partial t} - k_h \frac{\partial^2 T_s}{\partial z^2} = Q_{pc}^{subl} + Q_{pc}^{melt} + Q_{sw}. \quad \text{(LocalHeatTransfer)}$$

We have divided the phase change source term to distinguish the two possible phase change processes melt / freeze and sublimation / re-sublimation. The latter is omni-present in snow and is the one we want to include in the parameterization for the thermal conductivity. If we now define:

$$k_v = LD_p \frac{\partial p_v}{\partial T}, \quad \text{(ConducVapour)}$$

we see by comparison of Eqs. "ChangeIceMass" and "LocalHeatTransfer" that this definition of a thermal conductivity should represent the effect of vapour transport on the local temperature transport.
Now we note that the last term is given by the Clausius Clapeyron equation:

$$\frac{\partial p_v}{\partial T} = \frac{L}{T\Delta V}, \quad \text{(ConducVapour)}$$

where $\Delta V\ (m^3)$ is the volume change during phase change, which can be approximated by gas volume to give when using the ideal gas law:

$$k_v = LD_p \frac{Lp_v}{T^2 R_v}. \quad \text{(ConducVapourFinal)}$$

$R_v\ (\frac{m^2}{s^2 K})$ is the gas constant for water vapour. Finally we note that – if needed – we can replace the diffusion constant based on the partial pressure gradient by the diffusion constant based on vapour concentration, which is typically given from experimental work.

e) Final formulation of conductivity model
From the resistance for the solid ice phase given above and simply using Eqs. "ThermalResistance" and "ParallelResistances" we can then write for the thermal conductivity model:

$$k_h = \left(\frac{\pi^2 r_c k_i N}{32} + \frac{(k_v + k_a)k_i A_{ip}}{l_{is}(k_v + k_a) + l_p k_i} + \frac{k_p A_p}{l_p}\right)n, \quad \text{(FinalModel)}$$

where n is another geometrical factor given by Adams and Sato and Lehning et al. (2002a) from the number of grains intercepted per unit length. Note that Lehning et al. (2002a) also give the straightforward extension for the case of liquid water and derive approximations for the length and area scales needed in the formulation. The purpose of the derivation reformulated here is mainly to show how physical reasoning can be used to derive an explicit model for a complicated process such as heat transfer in snow.