Advection Diffusion Equation (ADE)

$$\frac{\partial C}{\partial t} + \vec{u} \cdot \vec{\nabla} C = D \nabla^2 C$$ linear in $\vec{u}$ and $\vec{C}$

in components, the equation becomes

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = D \nabla^2 C$$

$$\vec{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u \vec{e}_x + v \vec{e}_y + w \vec{e}_z$$

Comparison with the Navier-Stokes equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = \frac{\mu}{\rho} \nabla^2 \vec{u} + \dots$$

Non-linear! Much more complex!

in components: $$\frac{\partial \vec{u}}{\partial t} + u \frac{\partial \vec{u}}{\partial x} + v \frac{\partial \vec{u}}{\partial y} + w \frac{\partial \vec{u}}{\partial z} = \frac{\mu}{\rho} \nabla^2 \vec{u} + \dots$$

1D ADE, point source

→ examples: - tracer going down a river
- last year's exam

$$\frac{\partial C}{\partial t} + V_{adv} \frac{\partial C}{\partial x} = D \frac{\partial^2 C}{\partial x^2} \quad (*)$$

We set $\xi = x - V_{adv}\, t$
$\tau = t$

We have

$$\frac{\partial}{\partial t}\cdot = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}\cdot + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau}\cdot = -V_{adv}\frac{d}{d\xi}\cdot + \frac{d}{d\tau}\cdot$$

$$\frac{\partial}{\partial x}\cdot = \left(\frac{\partial x}{\partial \xi}\right)\frac{\partial}{\partial \xi}\cdot + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau}\cdot = \frac{\partial}{\partial \xi}$$

$$\frac{\partial^2}{\partial x^2}\cdot = \frac{\partial^2}{\partial \xi^2}$$

(*) thus becomes

$$-V_{adv}\frac{\partial}{\partial \xi}C + \frac{\partial}{\partial \tau}C + V_{adv}\frac{\partial}{\partial \xi}C = D\frac{\partial^2 C}{\partial \xi^2}$$

$$\Rightarrow \frac{\partial C}{\partial \tau} = D\frac{\partial^2 C}{\partial \xi^2} \qquad D.E.$$

Solution already known $C(\xi,\tau) = \frac{M}{A\sqrt{4\pi D\tau}}e^{-\xi^2/4D\tau}$

$$\Rightarrow C(x,t) = \frac{M}{A\sqrt{4\pi D t}}e^{-(x-V_{adv}t)^2/4Dt}$$

↳ transform in the other direction

I.C. $C(x,t=0) = \frac{M}{A}\delta(x)$ satisfied

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Lengthscales

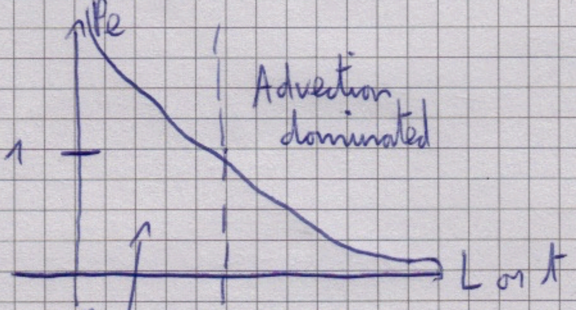
From the dimensional analysis, we know the dimensionless number

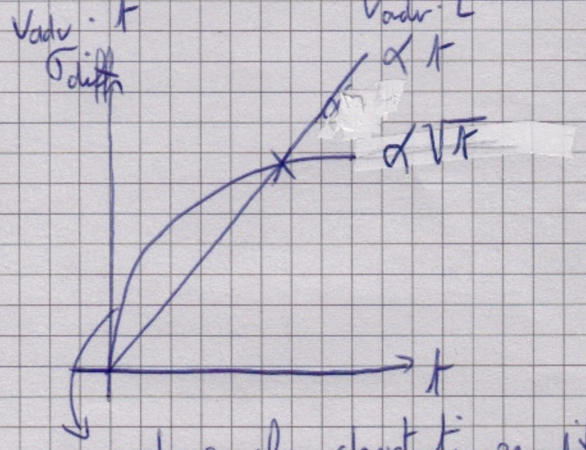
$$Pe := \frac{D}{v_{adv}^2\, t} \qquad \text{"Peclet number"}$$

⚠ sometimes the fraction is reversed in the definition of the Peclet number

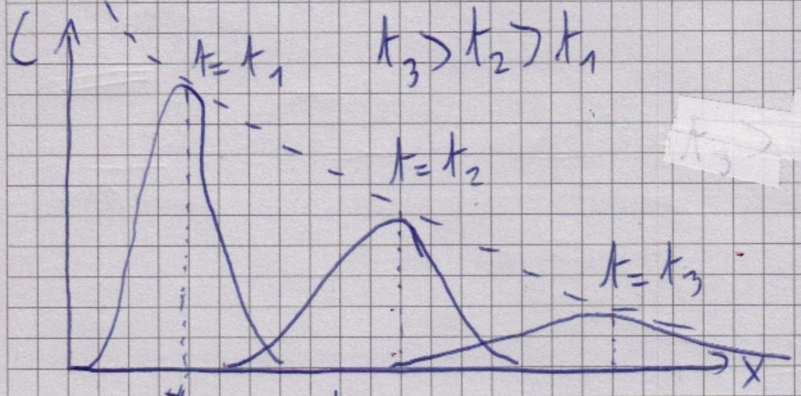
The Peclet number can also be written as $Pe = \frac{D}{v_{adv} \cdot L}$ with $L = v_{adv} \cdot t$.

$L \to$ distance downstream



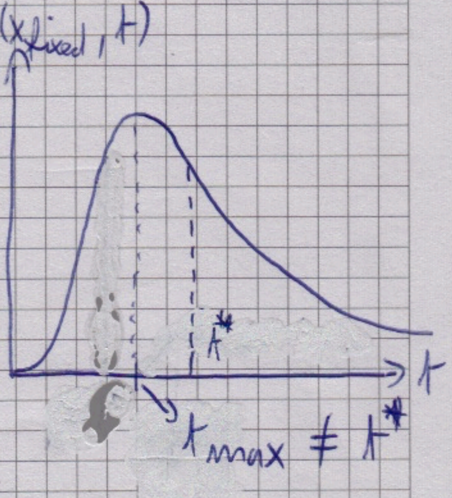


∞ slope, for short times diffusion always dominates



Max located at $x^* = v_{adv} \cdot t$, $C_{max} = \frac{M}{A\sqrt{4\pi D t}}$

$\boxed{\text{Fixed time}}$



Not symmetric!! $t^* = \frac{x_{fixed}}{v_{adv}}$

$\boxed{\text{Fixed position}}$