Environmental Transport Phenomena

Turbulent Dispersion in Rivers

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Turbulent Dispersion

I. Turbulent dispersion in river flows

II. Derivation of the 1-D advection-dispersion equation

III. Dye studies in rivers

IV. Examples

Reference: Book, Chapter 3

Turbulent Dispersion – Longitudinal Dispersion (Fischer, 1979)

- **Recall: Advection-diffusion equation for turbulent flows:**

$$\frac{\partial \overline{C}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{u}_i \; \overline{C}\right) = -\frac{\partial}{\partial x_i}\left[\overline{u_i' \; C'}\right] + \frac{\partial}{\partial x_i}\left[D_m \frac{\partial \overline{C}}{\partial x_i}\right]$$

$$\frac{\partial \overline{C}}{\partial t} + \overline{u}_i \frac{\partial \overline{C}}{\partial x_i} = \frac{\partial}{\partial x_i}\left[D_t \frac{\partial \overline{C}}{\partial x_i}\right]$$

$$\frac{\partial \overline{C}}{\partial t} + \overline{u}\frac{\partial \overline{C}}{\partial x} + \overline{v}\frac{\partial \overline{C}}{\partial y} + \overline{w}\frac{\partial \overline{C}}{\partial z} = \frac{\partial}{\partial x}\left[D_{t,x}\frac{\partial \overline{C}}{\partial x}\right] + \frac{\partial}{\partial y}\left[D_{t,y}\frac{\partial \overline{C}}{\partial y}\right] + \frac{\partial}{\partial z}\left[D_{t,z}\frac{\partial \overline{C}}{\partial z}\right]$$

- **In rivers,** it is common to do **1-D modeling:**

$$\frac{\partial \hat{C}}{\partial t} + \hat{u}\frac{\partial \hat{C}}{\partial x} = \frac{\partial}{\partial x}\left(D_L \frac{\partial \hat{C}}{\partial x}\right)$$

Advection-Dispersion Equation

('hat' indicates depth-averaged quantity)

Turbulent Dispersion – Longitudinal Dispersion (Fischer, 1979)

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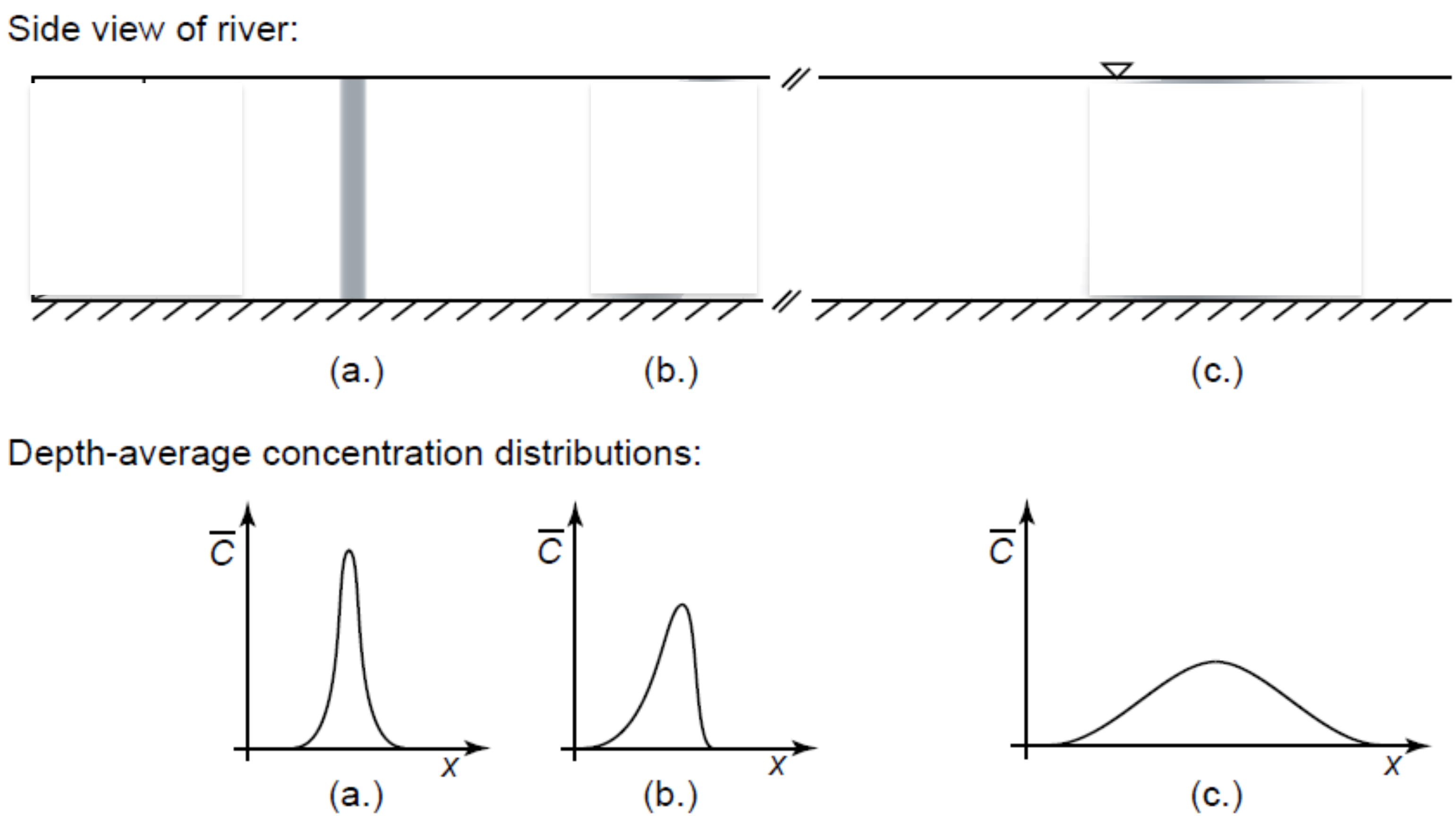
Advection-Dispersion Equation

('hat' indicates depth-averaged quantity)

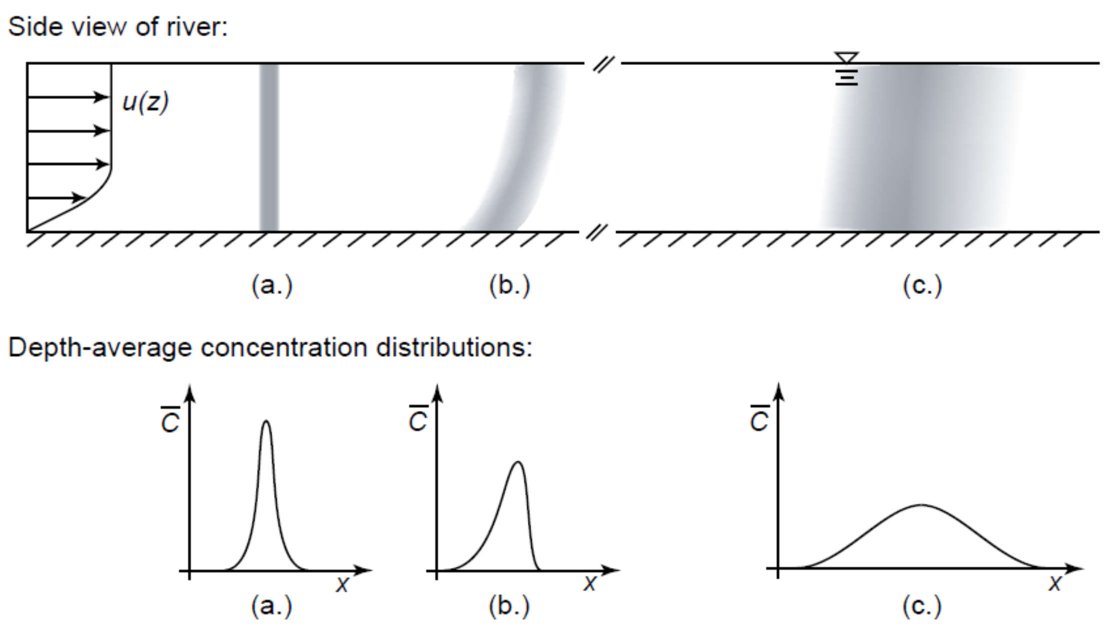
- **Goal:** Study the effect of velocity deviations in space due to non-uniform velocity (shear flow) on transport of contaminants



Turbulent Dispersion – Longitudinal Dispersion (Fischer, 1979)

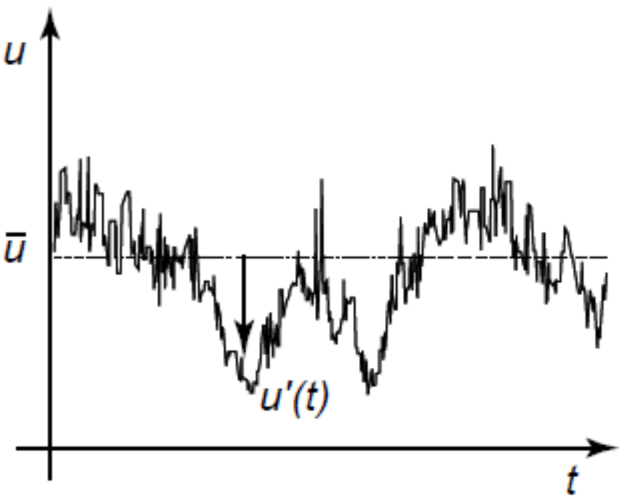
Injection of scalar (contaminant) → **Advection (non-uniform** due to shear) → **Vertical gradient** of concentration **C** at (b) → Net Turbulent diffusive flux

- **Dispersion** is the result of the **combined effects of Advection** (non-uniform) **and Vertical** (and lateral) **Diffusion**.



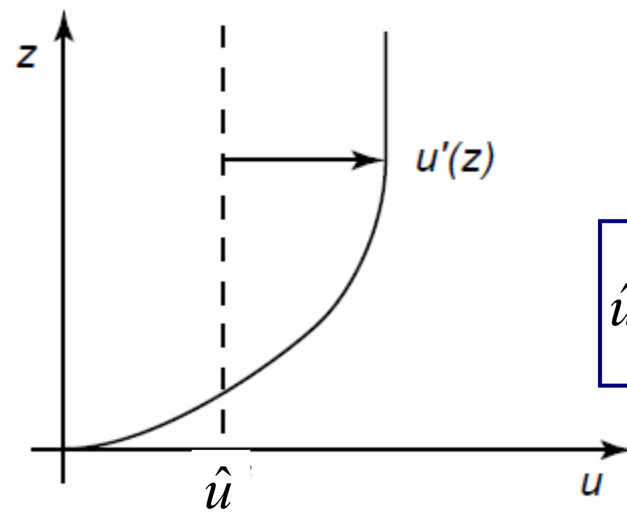
Turbulent Dispersion – Longitudinal Dispersion (Fischer, 1979)

(Recall: Reynolds decomposition)



$$u_i(t) = \bar{u}_i + u_i'(t)$$
$$C(t) = \bar{C} + C'(t)$$

(**New decomposition**)



$$\hat{u} = \frac{1}{h}\int_0^h u\,dz$$

$$u_i(x,z) = \hat{u}_i(x) + u_i'(x,z)$$
$$C(x,z) = \hat{C}(x) + C'(x,z)$$

Note: The new decomposition (based on **depth averaging**) is similar (but also quite different!) to Reynolds decomposition [Caution: It is better to use different notation]

Advection-Dispersion Equation *[derivation in Appendix - optional]*

Starting with the **Reynolds averaged equation**:

$$\frac{\partial \overline{C}}{\partial t} + \overline{u}_i \frac{\partial \overline{C}}{\partial x_i} = \frac{\partial}{\partial x_i}\left[D_t \frac{\partial \overline{C}}{\partial x_i} \right]$$

and applying depth averaging and some math, one can derive (appendix) the **depth-averaged governing equation**:

$$\frac{\partial \hat{C}}{\partial t} + \hat{u} \frac{\partial \hat{C}}{\partial x} = \frac{\partial}{\partial x}\left(D_L \frac{\partial \hat{C}}{\partial x} \right)$$

Advection-Dispersion Equation

[*Note: Since $D_L >> D_{tx}$, the diffusion term is neglected*]

$$D_L = -\frac{1}{h}\int_0^h u' \int_0^z \frac{1}{D_z} \int_0^z u' dz dz dz$$

$$D_L >> D_{t,x}$$

[*Note: Recall the discussion at the end of week 07*]

[*Note: Here we considered a very wide channel, but we could also include the effect of lateral gradients in cases where they are important –e.g., narrower rivers; in those cases, the equation above would also include D_y*]

Calculating the longitudinal dispersion coefficient

- **Analytical solutions to the integral (definition):**

* Turbulent flow in a **pipe** (of radius *a*): $D_L = 10.1\, a\, u_*$

* Turbulent flow **in very wide open channel** (using logarithmic velocity profile):

$$D_L = 5.93\, h\, u_*$$

Note: This is much larger than the turbulent diffusion defined earlier; recall:

$$D_{t,x} = 0.15\, h\, u_*$$

[*Note: This supports the assumption that* $D_L >> D_{t,x}$]

h =flow depth in a channel

Calculating the longitudinal dispersion coefficient

- **Numerical Integration**

The integral can be calculated numerically.

- **Engineering Estimates**

$$D_L = 0.011 \frac{\hat{u}^2 \; W^2}{d \; u_*}$$

W = average channel width

d = channel depth

Note: This empirical equation has been found to agree with observations **within a factor of 4 or so**. Deviations are due to factors such as recirculation zones and dead zones.

Calculating the longitudinal dispersion coefficient

- **Geomorphological Estimates** For straight rivers (Deng et al., 2001)

$$\frac{D_L}{hu_*} = \frac{0.15}{8\varepsilon_{to}}\left(\frac{W}{h}\right)^{5/3}\left(\frac{\hat{u}}{u_*}\right)$$

$$\varepsilon_{to} = 0.145 + \left(\frac{1}{3520}\right)\left(\frac{\hat{u}}{u_*}\right)\left(\frac{W}{h}\right)^{1.38}$$

- **Dye Studies (next section)**

Application: Dye Studies

<u>Goal:</u> To determine a river's flow rate and transport properties.

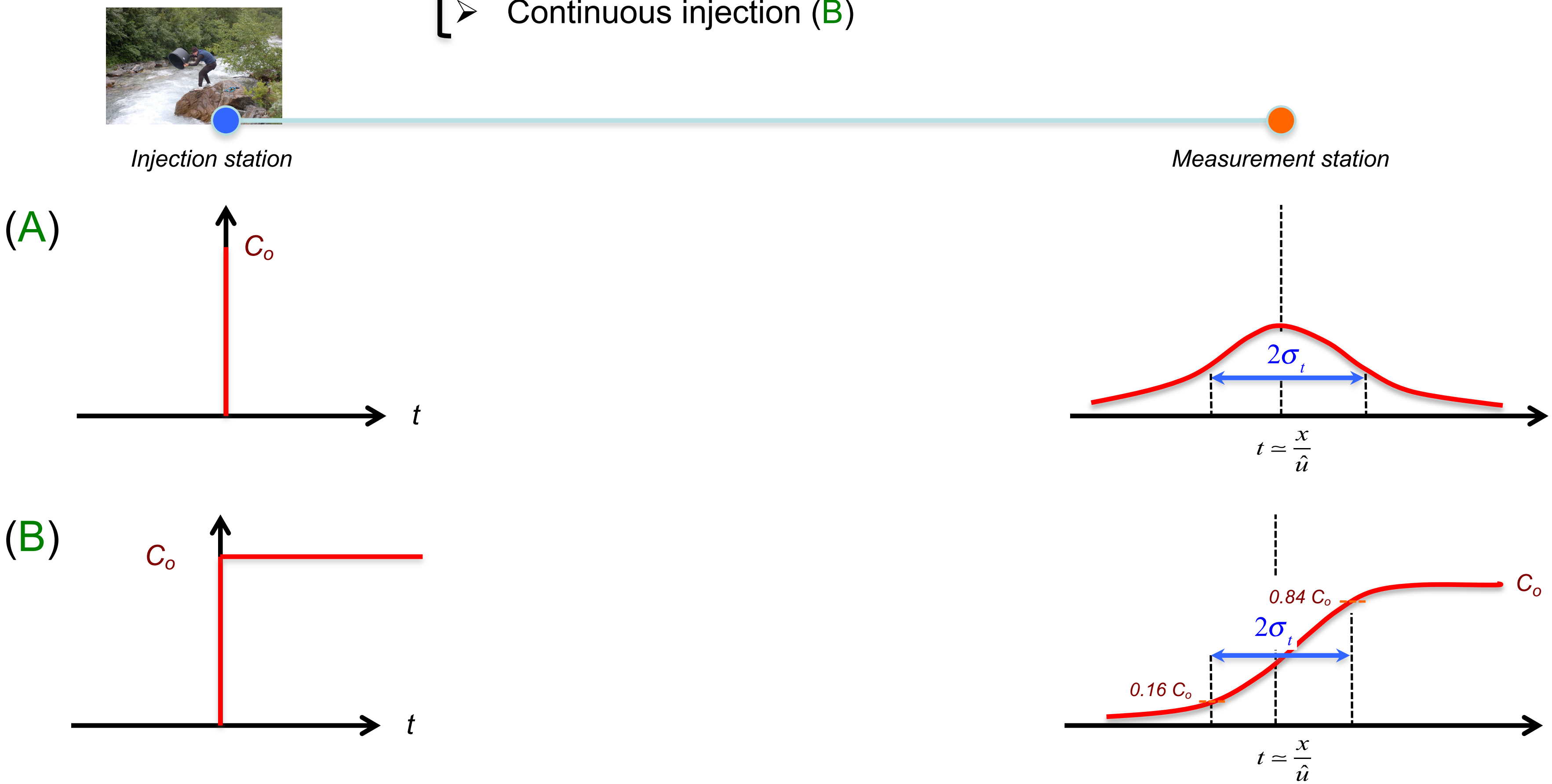
In particular:
- Mean advective velocity (**river flow rate**)
- **Effective longitudinal dispersion coefficient**

Two types of injections:
- Instantaneous
- Continuous

Application: Dye Studies

Two types of dye injection:

- Instantaneous injection (A)
- Continuous injection (B)



- From the measurements, we can get σ_t
- $\sigma_x \simeq \sigma_t \cdot \hat{u}$
- Finally, we can estimate D_L as: $$D_L \simeq \frac{\sigma_x^2}{2t} \simeq \frac{\sigma_t^2 \hat{u}^2}{2t}$$

Application: Dye Studies

- **<u>Continuous Injection (Example in the book)</u>**

Example: - River with rectangular cross-section: $h = 0.35 \ m$ $W = 10 \ m$

- Dropping leaves on surface one estimates $\hat{U} = 0.45 \ m/s$

- Slope: $S = 0.0005$

Using an engineering estimate: $D_L = 0.011 \dfrac{U^2 W^2}{d\sqrt{gdS}} = 15.4 \ m^2/s$

Application: Dye Studies

River Flow Rate (for the case of constant injection)**:**

$$Q_r = \frac{\dot{m}}{C_r}$$

$\dot{m} =$ Dye mass flow (g/s)

$C_r =$ Steady-state concentration (g/m^3)

Application: Dye Studies

Breakthrough curve (for the case of constant injection)**:**

Concentration measured at the (downstream) measurement station

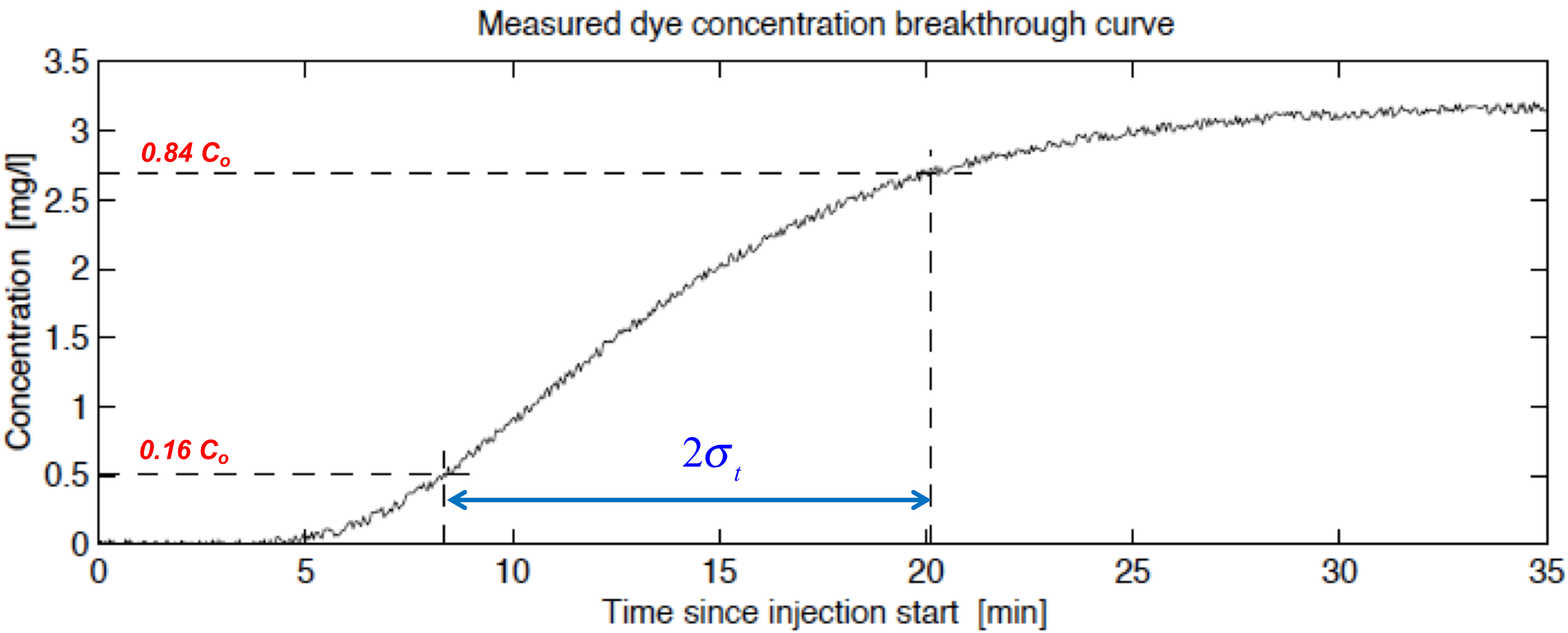
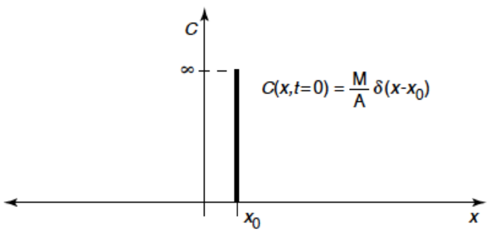
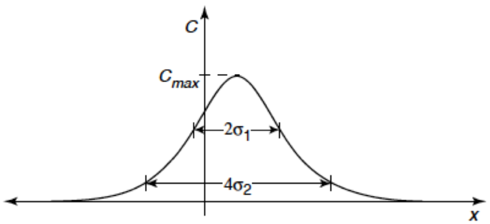


Table 2.1: Table of solutions to the diffusion equation

Schematic and Solution

Instantaneous point source, infinite domain





$$C(x,t) = \frac{M}{A\sqrt{4\pi Dt}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right]$$

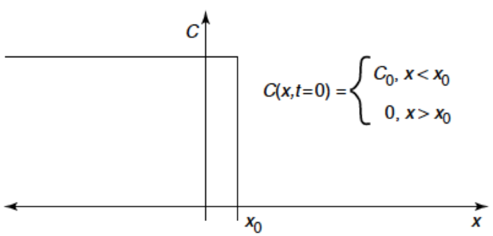
$$C_{max}(t) = \frac{M}{A\sqrt{4\pi Dt}}$$

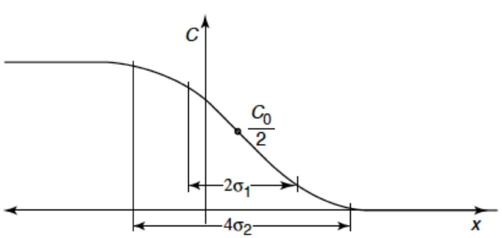
$$q_x(x,t) = \frac{M(x-x_0)}{2At\sqrt{4\pi Dt}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right]$$

Let $\sigma = \sqrt{2Dt}$ and $(2\sigma)^2 = 8Dt$.
For $x_0 = 0$:
$C(\pm\sigma, t) = 0.61C_{max}(t)$

Let $\sigma = \sqrt{2Dt}$ and $(4\sigma)^2 = 32Dt$.
For $x_0 = 0$:
$C(\pm 2\sigma, t) = 0.14C_{max}(t)$

Instantaneous distributed source, infinite domain





$$C(x,t) = \frac{C_0}{2}\left[1 - \text{erf}\left[\frac{(x-x_0)}{\sqrt{4Dt}}\right]\right]$$

$$C_{max}(t) = C_0$$

$$q_x(x,t) = \frac{C_0\sqrt{D}}{\sqrt{4\pi t}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right]$$

Let $\sigma = \sqrt{2Dt}$ and $(2\sigma)^2 = 8Dt$.
For $x_0 = 0$:
$C(+\sigma, t) = 0.16C_0$
$C(-\sigma, t) = 0.84C_0$

Let $\sigma = \sqrt{2Dt}$ and $(4\sigma)^2 = 32Dt$.
For $x_0 = 0$:
$C(+2\sigma, t) = 0.02C_0$
$C(-2\sigma, t) = 0.98C_0$

Application: Dye Studies – River Dispersion Coefficients

Longitudinal dispersion coefficient can also be computed from the breakthrough curve.

$$\sigma^2 = 2D_L\ t$$

The center of the dye front can be taken at: $C = 0.5C_0$

It passed the station at: $t = 12.94\ \text{min}$

One standard deviation to the left: $C = 0.16C_0$, $t = 8.35\ \text{min}$

One standard deviation to the right: $C = 0.84C_0$, $t = 20.12\ \text{min}$

Application: Dye Studies – River Dispersion Coefficients

We can compute:

$$\hat{u} = 0.45 \ \ m/s$$

$$2\sigma_t = 20.12 - 8.35 = 11.77 \ \ \text{min}$$

$$t = 8.35 + 11.77 / 2 = 14.24 \ \ \text{min}$$

$$\sigma = \hat{u} \ \ \sigma_t$$

$$D_L = \frac{\hat{u}^2 \sigma_t^2}{2t} = 14.8 \ \ \ m^2/s$$

Note: This is very similar to the engineering estimate computed earlier (15.4 m^2/s); but in many cases the differences are larger!

Application: Dye Studies – River Dispersion Coefficients

After estimating the diffusion coefficient, we are ready to use it to solve the advection-dispersion equation (recall: this is a 1-D problem):

$$\frac{\partial \hat{C}}{\partial t} + \hat{u}\frac{\partial \hat{C}}{\partial x} = \frac{\partial}{\partial x}\left(D_L \frac{\partial \hat{C}}{\partial x} \right)$$

Advection-Dispersion Equation

Application: Dye Studies – River Dispersion Coefficients

Example (Next Homework):

An **instantaneous point source**

Solution (recall the earlier lectures for derivation of this analytical solution for instantaneous point source):

$$\hat{C}(x,t) = \frac{M}{A\sqrt{4\pi D_L t}} \exp\left(-\frac{\left(x-\left(x_0+\hat{u}t\right)\right)^2}{4D_L t} \right)$$

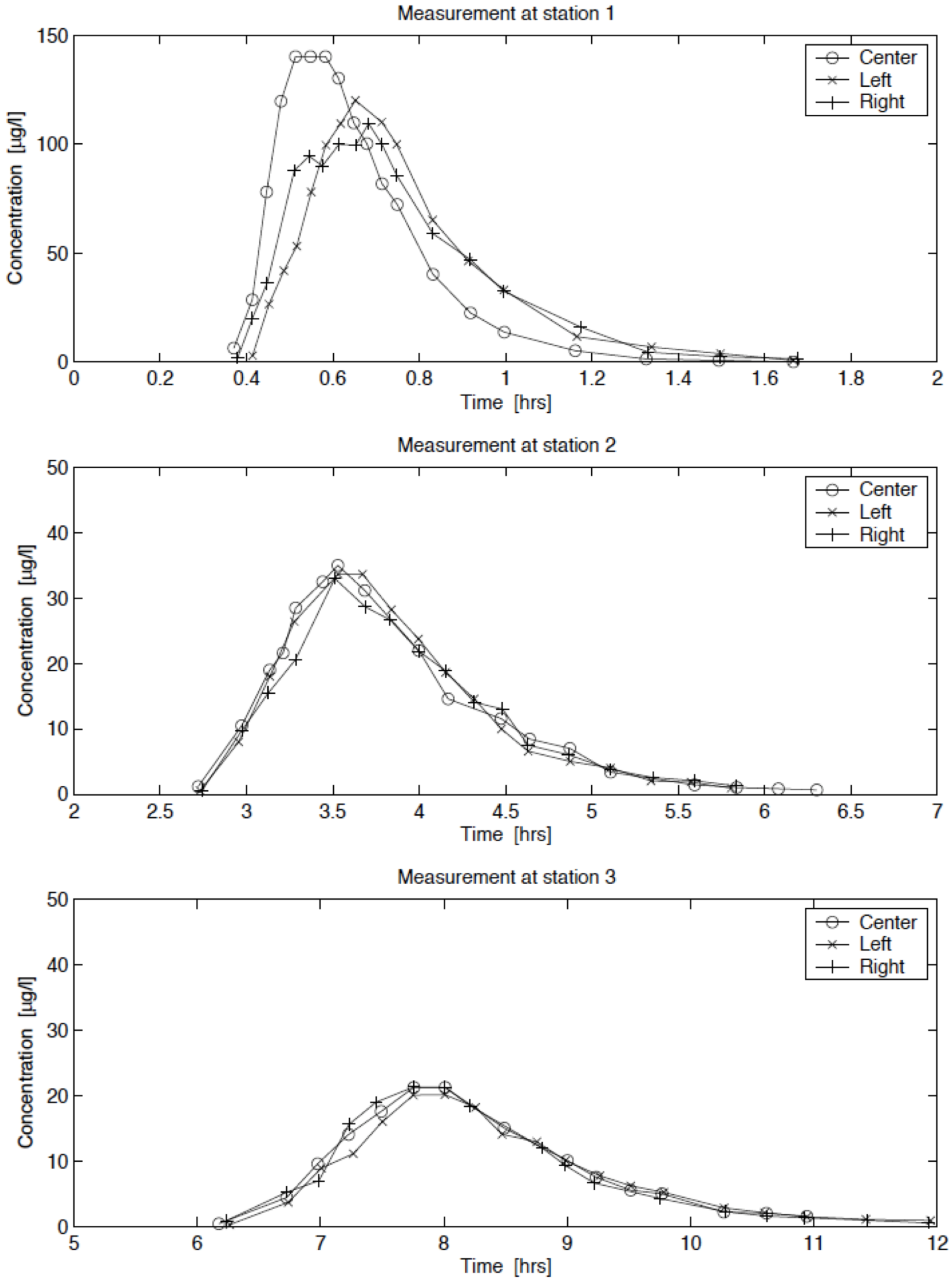
M : the mass of the tracer released in the channel

A : the cross-sectional area of the channel

x_0 : the injection point of the tracer

$\hat{u}$: the mean flow velocity

$\hat{u}t$: the distance traveled by the center of mass of the cloud at time t


Measurement at station 1
Concentration [μg/l]
Time [hrs]
Center
Left
Right
Measurement at station 2
Concentration [μg/l]
Time [hrs]
Center
Left
Right
Measurement at station 3
Concentration [μg/l]
Time [hrs]
Center
Left
Right

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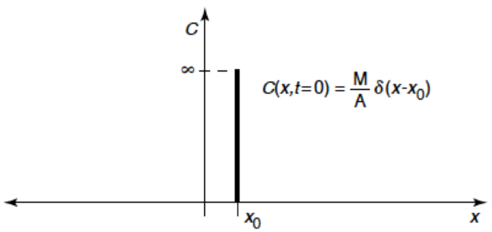

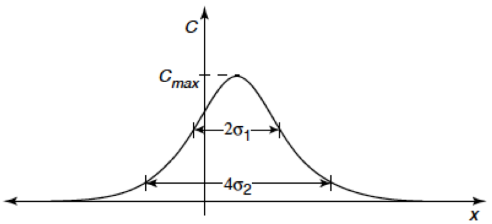

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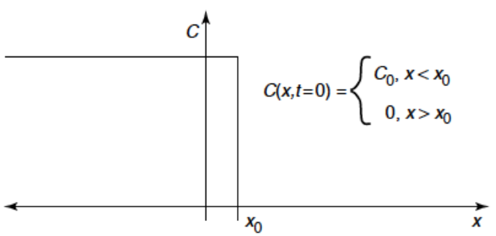

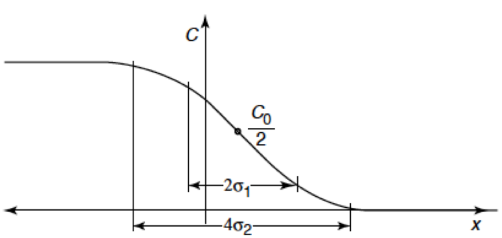

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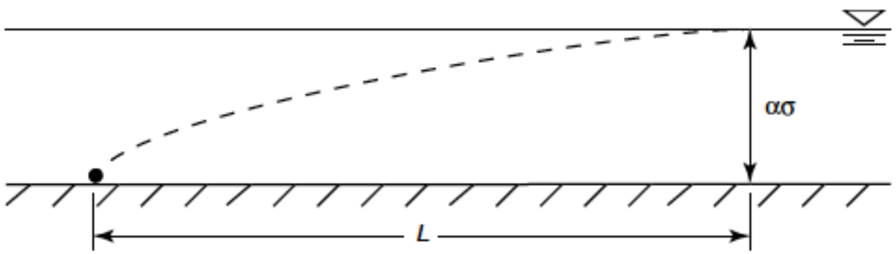
For $x_0 = 0$:
$C(+2\sigma, t) = 0.02 C_0$
$C(-2\sigma, t) = 0.98 C_0$

Example: Estimation of distance needed to have vertically well-mixed scalar

Example Box 3.2:
Vertical mixing in a river.

A factory waste stream is introduced through a lateral diffuser at the bed of a river, as shown in the following sketch.



At what distance downstream can the injection be considered as fully mixed in the vertical?

The assumption of "fully mixed" can be defined as the condition where concentration variations over the cross-section are below a threshold criteria. Since the vertical domain has two boundaries, we have to use an image-source solution similar to (2.47) to compute the concentration distribution. The results can be summarized by determining the appropriate value of α in the relationship

$$h = \alpha\sigma$$

where h is the depth and σ is the standard deviation of the concentration distribution. Fischer et al. (1979) suggest $\alpha = 2.5$.

For vertical mixing, we are interested in the vertical turbulent diffusion coefficient, so we can write

$$h = 2.5\sqrt{2D_{t,z}t}$$

where t is the time required to achieve vertical mixing. Over the time t, the plume travels downstream a distance $L = \overline{u}t$. We can also make the approximation $u_* = 0.1\overline{u}$. Substituting these relationships together with (3.19) gives

$$h = 2.5\sqrt{2 \cdot 0.067h(0.1\overline{u})L/\overline{u}}.$$

Solving for L gives

$$L = 12h.$$

Thus, a bottom or surface injection in a natural stream can be treated as fully vertically mixed after a distance of approximately 12 times the channel depth.

Appendix

Derivation of Advection-Dispersion Equation

Optional Reading Material

Derivation of Advection-Dispersion Equation *[optional]*

Note: Dye patch is introduced as a patch → $\frac{\partial C}{\partial y} = 0$

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}\left(D_x \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial z}\left(D_z \frac{\partial C}{\partial z}\right)$$

$D_x \quad D_z$ → Molecular/Turbulent diffusion coefficients for laminar/turbulent flow (if turbulent flow, then *u* and *C* would be time averaged).

Using decomposition:

$$u_i(x,z) = \hat{u}_i(x) + u_i'(x,z)$$

$$C(x,z) = \hat{C}(x) + C'(x,z)$$

$$\frac{\partial\left(\hat{C}+C'\right)}{\partial t} + \left(\hat{u}+u'\right)\frac{\partial\left(\hat{C}+C'\right)}{\partial x} = \frac{\partial}{\partial x}\left(D_x \frac{\partial\left(\hat{C}+C'\right)}{\partial x}\right) + \frac{\partial}{\partial z}\left(D_z \frac{\partial\left(\hat{C}+C'\right)}{\partial z}\right)$$

Note: Longitudinal dispersion >> Longitudinal diffusion (can neglect first term on the right side) [*Note: we are already using the result –this assumption will be shown more clearly later*]

Derivation of Advection-Dispersion Equation

Let us use this coordinate transformation (Lagrangian frame of reference: moving with mean advection velocity $\hat{u}$)

$$\xi = x - \hat{u}t$$
$$\tau = t$$
$$z = z$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial x} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial}{\partial \tau} - \hat{u}\frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial z} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z} = \frac{\partial}{\partial z}$$

Note: $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \hat{u}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial t} + \hat{u}\frac{\partial}{\partial x}$

Derivation of Advection-Dispersion Equation

$$\frac{\partial\left(\hat{C}+C'\right)}{\partial\tau}+\frac{\partial u'\left(\hat{C}+C'\right)}{\partial\xi}=\frac{\partial}{\partial z}\left(D_z\frac{\partial C'}{\partial z}\right)$$ [*]

Depth averaging: $\hat{C}=\frac{1}{h}\int_0^h C dz$

$$\frac{\partial\hat{C}}{\partial\tau}+\frac{\partial\widehat{u'C'}}{\partial\xi}=0$$ [**]

Depth-averaged governing equation → *Define 'dispersive flux':* $\hat{q}=\widehat{u'C'}$

Substracting: [*] –[**]

$$\frac{\partial C'}{\partial\tau}+u'\frac{\partial\hat{C}}{\partial\xi}+u'\frac{\partial C'}{\partial\xi}=\frac{\partial\widehat{u'C'}}{\partial\xi}+\frac{\partial}{\partial z}\left(D_z\frac{\partial C'}{\partial z}\right)$$

Governing equation for C' (concentration deviations from depth average)

Derivation of Advection-Dispersion Equation *[optional]*

Scale analysis at location (c)

$C' << \hat{C}$, and therefore:

$$u' \frac{\partial C'}{\partial \xi} << u' \frac{\partial \hat{C}}{\partial \xi}$$

$$\frac{\partial \widehat{u'C'}}{\partial \xi} << u' \frac{\partial \hat{C}}{\partial \xi}$$

$$\frac{\partial C'}{\partial \tau} + u' \frac{\partial \hat{C}}{\partial \xi} = \frac{\partial}{\partial z}\left(D_z \frac{\partial C'}{\partial z} \right)$$

Note: at point (c), vertical concentration deviations *C'* will reach a steady state (time invariant). Therefore, we can write:

$$u' \frac{\partial \hat{C}}{\partial \xi} = \frac{\partial}{\partial z}\left(D_z \frac{\partial C'}{\partial z} \right)$$

Derivation of Advection-Dispersion Equation *[optional]*

Integrating twice:

$$C'(z) = \frac{\partial \hat{C}}{\partial \xi} \int_0^z \frac{1}{D_z} \int_0^z u' dz dz$$

Recall depth-avg. equation:

$$\frac{\partial \hat{C}}{\partial \tau} + \frac{\partial \widehat{u'C'}}{\partial \xi} = 0$$

Define 'dispersive flux':

$$\hat{q} = \widehat{u'C'}$$

Using the definition of depth average:

$$\hat{q} = \frac{1}{h} \int_0^h u'C' dz = \widehat{u'C'}$$

Derivation of Advection-Dispersion Equation *[optional]*

$$\hat{q} = \frac{1}{h}\int_0^h u' \frac{\partial \hat{C}}{\partial \xi} \int_0^z \frac{1}{D_z} \int_0^z u' dz dz dz$$

$$\hat{q} = -D_L \frac{\partial \hat{C}}{\partial \xi}$$

$$D_L = -\frac{1}{h}\int_0^h u' \int_0^z \frac{1}{D_z} \int_0^z u' dz dz dz$$

Question: What happens with that equation if u'(z)=0 ? [hypothetical case of uniform velocity distribution]

Derivation of Advection-Dispersion Equation *[optional]*

Introducing into the depth-averaged governing equation:

$$\frac{\partial \hat{C}}{\partial \tau} + \frac{\partial \widehat{u'C'}}{\partial \xi} = 0 \quad \longrightarrow \quad \frac{\partial \hat{C}}{\partial \tau} = \frac{\partial}{\partial \xi}\left(D_L \frac{\partial \hat{C}}{\partial \xi} \right)$$

In the original coordinate system:

$$\frac{\partial \hat{C}}{\partial t} + \hat{u}\frac{\partial \hat{C}}{\partial x} = \frac{\partial}{\partial x}\left(D_L \frac{\partial \hat{C}}{\partial x} \right)$$

Advection-Dispersion Equation

$$D_L = -\frac{1}{h}\int_0^h u' \int_0^z \frac{1}{D_z}\int_0^z u' dz dz dz$$

$$D_{t,x} \ll D_L$$

[*Note: This supports the assumption made on page 23*]